Mathematical Reviews

Edited by

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R. P. Boas, Jr., *Executive Editor*

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Mathematical Reviews

FOUNDATIONS

Federici, Carlo.   On a law of duality in logic.   Univ. Nac. Colombia 14, 231–238 (1 plate) (1949).   (Spanish)

This is an expository article. The duality in question is that of the classical propositional algebra in truth-table form.   *H. B. Curry* (State College, Pa.).

Sobociński, Bolesław.   L'analyse de l'antinomie russellienne par Leśniewski.   Methodos 1, 94–107 (1949).

An analysis of the Russell paradox was made by Leśniewski twenty years ago or more. He claimed to have explained, not merely avoided, the paradox, in that he pointed out the erroneous assumption which would be present if the argument leading to the contradiction were formulated with sufficient explicitness. No full exposition of this analysis has ever been published. In this paper the author gives a few variants of such an explicit formulation, in such a way as to be as free as possible from technicalities. A continuation of the paper is to follow.   *H. B. Curry.*

Bourbaki, N.   Foundations of mathematics for the working mathematician.   J. Symbolic Logic 14, 1–8 (1949).

This is an address giving the point of view of Bourbaki and his collaborators on the logical foundations of mathematics. We are told that freedom from contradiction is an ideal not always attainable; the working mathematician should be content to formulate his rules so that eventual contradictions may always be traced to their sources. There follows a detailed description of a particular formal logical system, which is proposed as constituting the grammar of the mathematical language as actually used by mathematicians. This system is in terms of variables, denoted by small letters; the surrounding line, replaceable in practice by parentheses because of difficulties of typography; the three logical constants and, or, not; three mathematical signs $=$, ε, $|$, denoting identity, membership, and the relation between an ordered pair and its elements; and the quantifiers $\exists$ and $\forall$. There is given a set of rules for writing down well-formed formulas, which the author calls "relations." These rules, as well as other rules needed when abbreviations are introduced, must prescribe which arguments in a relation are free and which are bound.

Next are stated rules of substitution and rules of inference. The latter involve the notions of "synonymous" relations and "true" relations. A novel feature of the system is the introduction of relative notions by means of abbreviations; in particular a relation may be P-true, or true relative to a proof P. Likewise symbols for "relative" or "typical" quantifiers $\exists_H$ and $\forall_H$ are introduced; these quantifiers are relative to one of the hypotheses H of a proof P. Finally a set of axioms for set theory is described briefly. These include axioms for equality of individuals and ordered pairs, the sum axiom, the axiom of choice, axiom of infinity, and the pair-class axiom. A notation for descriptions is introduced. It is claimed that all present-day mathematics may be built up on these foundations, and that this will be demonstrated by means of the Bourbaki treatise, which is being published serially.   *O. Frink* (State College, Pa.).

Germansky, Baruch.   Supplement to my paper "Axioms of the natural numbers."   Riveon Lematematika 3, 8 (1949). (Hebrew)

The paper appeared in the same Riveon 1, 13 (1946); these Rev. 8, 126.

Nolfi, P.   Die Wahrscheinlichkeitstheorie im Lichte der dialektischen Philosophie.   Dialectica 3, 16–23 (1949).

Borel, Émile.   Probabilité et certitude.   Dialectica 3, 24–27 (1949).

Pólya, G.   Preliminary remarks on a logic of plausible inference.   Dialectica 3, 28–35 (1949).

After pointing out the duality in the notion of probability (limiting frequency, versus degree of rational belief), the author illustrates his view that the latter aspect may take the form of a probable implication. His examples are drawn from mathematical research (the "probability" of an unproved mathematical surmise), from puzzles ("this looks as if it would work"), and from practical experience ("this seems to be the right station to get off at"). The common feature of the probable implication is that as more and more necessary consequences of a statement are verified, its probability is increased. He does not wish to assume that in this connection "probability" means a number; but the notion is purposely left vague and intuitive.
                    B. O. Koopman (New York, N. Y.).

Gini, Corrado.   Concept et mesure de la probabilité.   Dialectica 3, 36–54 (1949).

Lévy, Paul.   Les fondements du calcul des probabilités.   Dialectica 3, 55–64 (1949).

de Finetti, B.   Le vrai et le probable.   Dialectica 3, 78–92 (1949).

Baptist, J. H.   Le raisonnement probabilitaire.   Dialectica 3, 93–103 (1949).

Bartlett, M. S.   Probability in logic, mathematics and science.   Dialectica 3, 104–113 (1949).

Féraud, L.   Induction amplifiante et inférence statistique.   Dialectica 3, 127–152 (1949).

Hagstroem, K.-G.   Connaissance et stochastique.   Dialectica 3, 153–172 (1949).

Pages 153–161 analyse the impact of the statistical way of thinking on epistemology. The remainder of the paper is devoted to a mathematical appendix, treating topics such as distribution functions, Fourier transforms, plane waves, the uncertainty principle.   *W. Feller* (Ithaca, N. Y.).

ALGEBRA

Kaluza, Th., jun. Über einige asymptotische und exakte Formeln für die Anzahlen von diskordanten Permutationen. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig 1946, no. 2, i+16 pp. (1946).

The main result proved is that the number of permutations discordant with each of the rows of an n by k Latin rectangle is asymptotically $n!e^{-k}$. This is included in the result of Erdös and Kaplansky [Amer. J. Math. 68, 230–236 (1946); these Rev. 7, 407]. An extensive treatment for the case $k=2$ reaches results like those given by Touchard [C. R. Acad. Sci. Paris 198, 631–633 (1934)].

         J. Riordan (New York, N. Y.).

Lévy, Paul. Sur une classe remarquable de permutations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 361–377 (1949).

A permutation Q_n of the elements $(1, \cdots, n)$ is defined in the following way. Take a hand of n cards and put the first card on the table, the second at the end of the hand, the third on the table, and so alternately until all cards are on the table: their order defines Q_n. The author studies various properties of Q_n, in particular the cycles, their length, their structure in dependence on n, etc. Space does not permit the reproduction of all the definitions without which the theorems cannot be stated. The paper is self contained, but generalizes and improves on previous results [C. R. Acad. Sci. Paris 227, 422–423, 578–579 (1948); 228, 1089–1090 (1949); these Rev. 10, 177, 130, 500].   *W. Feller.*

Pizá, Pedro A. Tables de polynomes. Mathesis 58, 159–163 (1949).

The author defines two sets of polynomials $P_c^{(n)}$ and $R_c^{(n)}$, with positive integral indices n and c, by the following recursion formulas:

$$P_c^{(n)} = (cx+y)P_c^{(n-1)} + P_{c-1}^{(n-1)}, \quad P_0^{(0)}=1, \quad P_c^{(n)}=0 \quad (c>n);$$
$$R_c^{(n)} = (cx+y)R_c^{(n-1)} + [(n-c+1)x-y]R_{c-1}^{(n-1)}, \quad R_0^{(0)}=1.$$

They are closely related to certain sets of numbers that the author has discussed elsewhere [see Arithmetical Essays, Santurce, P. R., 1948; these Rev. 10, 508]. Various properties of these polynomials are stated without proof and the reader is urged to send in proofs for them. Some of these are summation formulas, of which the following is typical:

$$\sum_{r=0}^{n}(rx+y)^n = \sum_{c=0}^{n}\binom{a+n-c+1}{n+1}R_c^{(n)}.$$

       H. W. Brinkmann (Swarthmore, Pa.).

Tenca, Luigi. Sulle radici n-esime primitive dell'unità. Period. Mat. (4) 27, 104–108 (1949).

Let $a_1, \cdots, a_n$ be n real numbers and $[i_1 \cdots i_n]$ the polynomial of degree n in which the coefficient of x^j is the a with subscript i_j. System $U[i_1 \cdots i_n]$ consists of the n equations got by subjecting the indices in $[i_1 \cdots i_n]$ to cyclic permutation. System $V[i_1 \cdots i_n]$ consists of the $\varphi(n)$ systems got by letting the indices in $U[i_1 \cdots i_n]$ assume all arrangements $j_1 \cdots j_n$ such that permutation $(j_1 \cdots j_n)$ generates the same cyclic group as does $(i_1 \cdots i_n)$. The circulant $D[i_1 \cdots i_n]$ whose first row consists of the coefficients of $[i_1 \cdots i_n]$ is the resultant of x^n-1 and any equation of $V[i_1 \cdots i_n]$. Suppose $D[i_1 \cdots i_n]=0$ and $D[k_1 \cdots k_n] \neq 0$ for $[k_1 \cdots k_n]=0$ not in $V[i_1 \cdots i_n]$. Denote by ϵ a root common to $[i_1 \cdots i_n]$ and x^n-1. The paper shows that: (α) ϵ is a root of $U[i_1 \cdots i_n]$; (β) the complex conjugate of ϵ is a root of $U[i_1 \cdots i_n]$; (γ) ϵ is a

root of $U[i_1i_n \cdots i_2]$ which is distinct from $U[i_1 \cdots i_n]$ if $2<n$ but is in $V[i_1 \cdots i_n]$; (δ) ϵ is a primitive root of x^n-1; (θ) if each $U[j_1 \cdots j_n]$ in $V[i_1 \cdots i_n]$ is satisfied by exactly one ϵ and its conjugate, the U's can be paired so that each pair has in common exactly one ϵ and its conjugate. If one $[j_1 \cdots j_n]$ is selected from each pair of U's, the product of the corresponding $D[j_1 \cdots j_n]$ is a circulant. If the first row of the circulant is $b_1 \cdots b_n$, all the primitive roots of x^n-1 are roots of $b_1+b_2x+\cdots+b_nx^{n-1}$. The author's statement (η) seems to be implied by his (γ) and the condition in (θ) seems to insure that at most two ϵ's are roots of $[i_1 \cdots i_n]$ rather than what the author says.   *J. M. Thomas.*

van der Corput, J. G. Sur les fonctions symétriques. Math. Centrum Amsterdam, Scriptum no. 3, 17 pp. (1949).

The author studies properties of certain differential operators which can be used to discover identities between symmetric functions. Let $\{a_i\}$ be the elementary symmetric functions of the m variables z_i. A summation $\sum z_1^{k_1} \cdots z_m^{k_m}$ carried out over all the permutations of the nonnegative integers k_m is called simple. If the k_i are arranged so that $k_1, \cdots, k_r>0$ but $k_{r+1}=\cdots=k_n=0$ the sum is denoted by $\{k_1, \cdots, k_r\}$. Assume the numbers $k_1, \cdots, k_r$ fall into u disjoint sets (containing $j_1, \cdots, j_u$ elements) such that all the k's in a particular set are equal. Consider the differential operator

$$[k_1, \cdots, k_r] = \frac{1}{j_1! \cdots j_u!} \sum_{\mu_1=k_1}^{m} \cdots \sum_{\mu_r=k_r}^{m} a_{\mu_1-k_1} \cdots a_{\mu_r-k_r} \frac{\partial^r}{\partial a_{\mu_1} \cdots \partial a_{\mu_r}}.$$

Here the $k_i \leq m$ are positive integers and $a_0=1$. This operator can be applied to rational functions in a_i and to simple sums since the latter can be expressed in terms of the a_i. The operator $A_k = [1, 1, \cdots, 1]$ (containing k units) is called k-cancelling since $A_k\{k_1, \cdots, k_r\}=0$ if k is different from all the $k_1, \cdots, k_r$; if k occurs among the $k_1, \cdots, k_r$ the operator replaces r by $r-1$ and suppresses one k. The following theorem is also established: an identity between the a_i and some simple sums remains valid if the a_i are replaced by A_i and each sum $\{k_1, \cdots, k_r\}$ by the operator $[k_1, \cdots, k_r]$.   *O. Todd-Taussky* (Washington, D. C.).

Sagastume Berra, Alberto E. On the theory of symmetric functions. An. Soc. Ci. Argentina 147, 235–253 (1949). (Spanish)

A symmetric polynomial in n variables $x_1, \cdots, x_n$ is a linear combination of "simple" symmetric functions of the form $\sum x_1^{h_1} \cdots x_n^{h_n}$, for which the author uses the notation $(h_1, \cdots, h_n)$. He also introduces symmetric functions of the "second order" by means of a matrix (h_{ij}) of nonnegative integers, such a function being defined as the sum $\sum_{(i)} \alpha_{(i)}(h_{1i_1}, \cdots, h_{ni_n})$, where $(i)=i_1, \cdots, i_n$ is a permutation of $1, 2, \cdots, n$ and the coefficient $\alpha_{(i)}$ is determined according to a certain rule; the summation is carried out over all the permutations (i). It turns out that the product of two simple symmetric functions is always expressible as a homogeneous function of the second order, and conversely. Thus these functions of the second order are useful when explicit expressions of symmetric functions in terms of the elementary symmetric functions are to be found. As an example a method is given for expressing the powers of an elementary symmetric function in terms of the elementary symmetric functions. The author also makes use of what he

calls the "dual" of a symmetric function; this is essentially the corresponding symmetric function of the reciprocals $1/x_1, \cdots, 1/x_n$ multiplied by a sufficiently high power of $x_1 x_2 \cdots x_n$. It is clear how the expression, in terms of elementary symmetric functions, can be found for the dual of a symmetric function if the corresponding expression for the original function is known. *H. W. Brinkmann.*

Hua, Loo-Keng. A theorem on matrices and its application to Grassmann space. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5, 150–181 (1948).

Let Z be an n by m matrix over a field Φ, σ an automorphism of Φ, P and Q nonsingular square matrices and R a fixed n by m matrix. Call two matrices coherent if the rank of their difference is one. In this paper it is proved that a one-to-one mapping $Z \rightarrow Z_1$ which leaves the coherence invariant is of the form (*) $Z_1 = PZ^\sigma Q + R$ for $1 < n < m$ and is either this or (**) $Z_1 = P(Z')^\sigma Q + R$, with the prime denoting the transpose, for $1 < n = m$. For $1 = n < m$, the theorem holds if the hypothesis is strengthened to require lines of the affine m-space to go into lines and incidence to be invariant. The method of proof is based on constructions involving the meet and join of linear subspaces of the space of matrices and thus differs from that used by the author to obtain similar results for symmetric matrices [Ann. of Math. (2) **50**, 8–31 (1949); these Rev. **10**, 424]. [Despite the later date of publication, these results were obtained earlier than those under review.]

In the second part of the paper a projective geometry of n by m matrices is developed. If $W = (X, Y)$ is of rank n, where X is n by m and Y is n by n, then QW for Q arbitrary nonsingular are homogeneous coordinates of a point. If Y is nonsingular, $Z = Y^{-1}X$ is the nonhomogeneous coordinate of the point, which is then called finite. Two points W and W_1 are called coherent if the rank of $\binom{W}{W_1}$ is $n+1$. The main theorem proved is: any one-to-one mapping carrying the projective space of n by m matrices onto itself and preserving coherence is of the form $W^* = QWP$ with Q and P square and nonsingular when $1 < n < m$. For $1 < n = m$, it is necessary to adjoin to the group another transformation, which in nonhomogeneous form can be $Z \rightarrow Z'$. *W. Givens* (Knoxville, Tenn.).

Ponting, F. W., and Potter, H. S. A. The volume of orthogonal and unitary space. Quart. J. Math., Oxford Ser. 20, 146–154 (1949).

The set of all n-rowed orthogonal matrices determines a $\frac{1}{2}n(n-1)$-dimensional manifold M_0 in the n^2-dimensional Euclidean space R, the elements of a matrix providing the coordinates of a point. The authors give another proof [cf. A. Hurwitz, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1897, 71–90] that the volume of M_0 is $2^{n(n+1)/4}\prod_{r=1}^n \pi^{r/2}/\Gamma(\frac{1}{2}r)$, where the metric used is the one induced by the metric in R. Likewise they find that the volume of the set of all n-rowed unitary matrices (in $2n^2$-dimensional space) is $2^{n(n+1)/2}\prod_{r=1}^n \pi^r/\Gamma(r)$. The proofs involve essentially matrix manipulations. *L. Tornheim* (Ann Arbor, Mich.).

Makar, Raouf H. The infinite matrix series $\sum A^r$ and the field P_A. Proc. Math. Phys. Soc. Egypt 3, 57–63 (1948).

An infinite matrix A, with complex elements, is self-associative if all finite powers of A exist and all products of such powers are associative. The matrix A is algebraic if it is self-associative and satisfies a scalar algebraic equation. The author proves that if A is algebraic and satisfies an equation $k_0 I + k_1 X + \cdots + k_n X^n = 0$, where $\sum_{i=0}^n k_i \neq 0$, then

a necessary and sufficient condition that the infinite series $\sum A^n$ converge is that $\lim_{N \to \infty} A^N$ exist. An explicit formula is obtained, in terms of the coefficients k_i, for the sum of the series if it converges. The second part of the paper is concerned with a number of elementary properties of the set P_A of matrices which can be expressed as a polynomial in a self-associative matrix A. *N. H. McCoy.*

Abstract Algebra

Ribenboim, Paulo. Characterization of the sup-complement in a distributive lattice with last element. Summa Brasil. Math. 2, no. 4, 43–49 (1949).

Theorem: if R is a distributive lattice with greatest element I, then R is "Brouwerian" (that is, distributive and join-complemented), with s as join-complementation, if there exists $s(x)$ such that, for all x in R, $x \cup s(x) = I$, $ss(x) \subset x$, $s(x \cap y) = s(x) \cup s(y)$, $ss(x \cup y) = ss(x) \cup ss(y)$. An example is given of a distributive lattice with units which is not join-complemented. *P. M. Whitman.*

Foster, Alfred L. On the permutational representation of general sets of operations by partition lattices. Trans. Amer. Math. Soc. 66, 366–388 (1949).

Let $\mathfrak{A}$ be a set of partitions (homomorphic with respect to a set of operations on U, including a set Ω of monotations (=unary operations)) of a set U. If D is the intersection of cells A_i, one from each member of $\mathfrak{A}$, then $\omega(D) = \prod \omega(A_i)$ (referring to the cellular monotations induced by a monotation $\omega \epsilon \Omega$). A monotation ω is schlicht if $x \neq y$ implies $\omega(x) \neq \omega(y)$, and is permutational if also $\omega(x) = y$ is always solvable; a partition is Ω-schlicht or Ω-permutational if each cellular monotation induced by the $\omega \epsilon \Omega$ is schlicht or permutational. The greatest lower bound of a set of Ω-schlicht partitions of U is again Ω-schlicht, but not so for least upper bound. A set $\mathfrak{A}'$ of Ω-schlicht partitions is said to satisfy the target condition if, for each $\omega \epsilon \Omega$, $\prod A_i = 0$ implies $\prod \omega(A_i) = 0$, where exactly one cell A_i is chosen from each partition belonging to $\mathfrak{A}'$. This condition is necessary and sufficient that the greatest lower bound of a set of Ω-permutational partitions is again Ω-permutational (if it holds for the set $\mathfrak{P}$ of all Ω-permutational partitions, then this greatest lower bound is called the atomic partition), and then the least upper bound is also Ω-permutational. In a finite U, $\mathfrak{P}$ is identical with the set of all Ω-schlicht partitions and is a lattice, namely the lattice of all Ω-partitions which have the atomic partition as a refinement. Atomic partitions are constructed in certain cases. Homomorphisms of a set with monotations, and their invariants, are considered. *P. M. Whitman* (Silver Spring, Md.).

Jordan, P. Zur Axiomatik der Verknüpfungsbereiche. Abh. Math. Sem. Univ. Hamburg 16, 54–70 (1949).

The author studies subsets of algebras closed under the operations of forming subalgebras, or homomorphic images (Darstellungen), or both. Since these operations are unary, such "closed" subsets form rings in the set of "all" algebras; hence distributive lattices. The duality between being closed under these operations, and satisfying given sets of laws, is discussed; in particular, laws of the type "A implies B or C" are studied. Further developments are promised. *G. Birkhoff* (Cambridge, Mass.).

Jordan, P. Zum Dedekindschen Axiom in der Theorie der Verbände. Abh. Math. Sem. Univ. Hamburg **16**, 71–73 (1949).

The author gives a short new proof of von Neumann's theorem, that three elements a, b, c of a modular lattice generate a distributive lattice if $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ or dually, and notes that any lattice having these properties is modular.   *G. Birkhoff* (Cambridge, Mass.).

Sikorski, Roman. A theorem on extension of homomorphisms. Ann. Soc. Polon. Math. **21** (1948), 332–335 (1949).

The author shows by transfinite induction that every homomorphism of a subalgebra of a Boolean algebra B into a complete Boolean algebra C can be extended to a homomorphism of B into C. One immediate consequence is the theorem of M. H. Stone that every ideal in a Boolean algebra can be extended to be a prime ideal.   *O. Frink.*

Hebroni, Pessach. On the solution of certain linear equations in the abstract ring. Riveon Lematematika **2**, 56–59 (1948). (Hebrew)

Let R be a ring with unit element containing the complex numbers. The author studies the linear equation in x, $f(a)x = c$, where $a \varepsilon R$, $c \varepsilon R$ and $f(a)$ is a polynomial in a with complex numbers as coefficients.   *A. Dvoretzky.*

Uzkov, A. I. On rings of quotients of commutative rings. Amer. Math. Soc. Translation no. 3, 5 pp. (1949).

Translated from Mat. Sbornik N.S. **22**(64), 439–441 (1948); these Rev. **10**, 97.

Zawrotsky, A. Some generalizations of the concept of field. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **10**, 171–191 (1946). (Spanish)

The author observes that there is difficulty in extending the sequence of operations addition—multiplication—exponentiation, because the last is not commutative and so lacks symmetry. Likewise, the definition of field is not quite symmetric since 0 must be excluded from the multiplicative group. The author seeks generalizations which avoid these inelegancies. It is postulated that one deals with a field F having a nontrivial function $L(x)$ such that $L(xy) = L(x) + L(y)$ for x and y in the multiplicative group of F. Then for $b \varepsilon F$ ($b \neq 0, 1$) there exists an $L(x)$, denoted by $L_b(x)$, with $L_b(1) = 0$, $L_b(b) = 1$. Postulate: for $x \varepsilon F$, there exists y such that $L_b(y) = x$. Definition:

$$L_b{}^n(x) = L_b[L_b{}^{n-1}(x)]; \quad S(x, y; 0) = x + y;$$
$$L_b{}^n[S(x, y; n)] = L_b{}^n(x) + L_b{}^n(y).$$

Theorems: $xy = S(x, y; 1)$; $x^y = S[x, L_b{}^{-1}(y); 2]$; $S(x, y; n) \varepsilon F$; there exists I_n such that $S(x, I_n; n) = x$ for $n > 0$.

If F is a complete field, the author postulates that $I_{-n} = L_b{}^n(0)$ exists for $n > 0$. Let F' denote F with $I_{-1}, I_{-2}, \cdots$ adjoined. Theorem: F' excluding $I_{n-1}, I_{n-2}, \cdots$ is a field with respect to the operations $S(x, y; n)$ and $S(x, y; n+1)$. The author considers in a similar way some properties of finite fields, generalizing the "irregular fields" of Bell [Algebraic Arithmetic, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927]. He sketches a method of replacing the concept of $L(x)$ by permutations, leading to generalized irregular fields of any finite order. Several explicit tables are appended.   *P. M. Whitman* (Silver Spring, Md.).

Serre, Jean-Pierre. Extensions de corps ordonnés. C. R. Acad. Sci. Paris **229**, 576–577 (1949).

Let L be an extension of the ordered field K. A necessary and sufficient condition that the ordering of K can be extended to an ordering of L is that $\sum p_i x_i^2 = 0$, where each p_i is a positive element of K and $x_i \varepsilon L$, implies that $x_i = 0$ for each i. The necessity is obvious, and the sufficiency is shown by the following method. Let P be a set of nonzero elements of L which contains all elements px^2, where p is a positive element of K and $x \varepsilon L$, and such that $P + P \subseteq P$, $P \cdot P \subseteq P$. If we now choose as the positive elements of L the elements of a maximal one among such sets P, this leads to an ordering of L which is an extension of the ordering of K. As a special case, one gets the result of Artin and Schreier [Abh. Math. Sem. Hamburg Univ. **5**, 85–115 (1926)] that a field can be ordered if and only if $\sum x_i^2 = 0$ implies that each $x_i = 0$. The author states some other related results, among them the following: if $f(x)$ is an irreducible polynomial over the ordered field K which changes sign on K, the field obtained by adjoining to K a root of $f(x)$ can be ordered in such a way as to extend the ordering of K.   *N. H. McCoy* (Northampton, Mass.).

✶ Deuring, Max. Algebraische Funktionenkörper und algebraische Geometrie. Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 149–162. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

Albert, A. A. Absolute-valued algebraic algebras. Bull. Amer. Math. Soc. **55**, 763–768 (1949).

A nonassociative (possibly infinite-dimensional) algebra is called algebraic in case each element generates a subalgebra of finite dimension. By proving the finite-dimensionality of any absolute-valued real algebraic algebra with unity quantity, the author extends to algebraic algebras his previous results for the finite-dimensional case [Ann. of Math. (2) **48**, 495–501 (1947); these Rev. **8**, 561]; namely, that any such algebra is either the field R of all real numbers, the field C of all complex numbers, the real quaternion algebra Q, or the real Cayley algebra D. Incidental to the proof of the chief result is the theorem that every alternative algebra with unity quantity over a field F of characteristic not two, having the property that each non-scalar element generates a quadratic field over F, is of finite dimension (necessarily 1, 2, 4, or 8) over F.   *R. D. Schafer* (Philadelphia, Pa.).

Albert, A. A. A note of correction. Bull. Amer. Math. Soc. **55**, 1191 (1949).

The existence of a two-dimensional absolute-valued real algebra without unity quantity draws attention to an error of omission in theorem 3 of the author's paper [Ann. of Math. (2) **48**, 495–501 (1947); these Rev. **8**, 561]. This error induces a corresponding one in theorem 3 of the paper reviewed above. None of the author's results on algebras with unity quantity are affected by this omission.   *R. D. Schafer* (Philadelphia, Pa.).

Jacobson, N. Derivation algebras and multiplication algebras of semi-simple Jordan algebras. Ann. of Math. (2) **50**, 866–874 (1949).

An algebra A (associative or not) is called right semi-simple if it is a direct sum of minimal right ideals none of which is annihilated by A to the right. The Lie algebra L of linear transformations of A which is generated by the

right multiplications by elements of A is called the enveloping algebra of the right multiplications of A. It is proved (theorem 1) that, if A is semi-simple over a field of characteristic 0 and has a unit element, then every derivation of A is of the form $U - L_u$, where U is in L and L_u is the left multiplication by $u = U(1)$. Similar definitions and statements are made for algebras which are left semi-simple or two-sidedly semi-simple. This theorem is applied to the study of derivations of semi-simple Jordan algebras A. In that case, the bracket of two multiplications of A is a derivation; these derivations, together with their linear combinations, are called inner derivations. Every element of the algebra L defined above is the sum of a multiplication and an inner derivation. If A is semi-simple and the basic field of characteristic 0, then it is proved (theorem 2) that every derivation is an inner derivation.

Making use of this result, the author determines the derivation algebras of the simple Jordan algebras of the three "big" classes [another determination of these derivation algebras by F. D. Jacobson and N. Jacobson has appeared in Trans. Amer. Math. Soc. 65, 141–169 (1949); these Rev. 10, 588]. It turns out that every nonexceptional simple Lie algebra may be represented in this manner. The enveloping Lie algebras of the same Jordan algebras are also explicitly determined.

C. Chevalley (New York, N. Y.).

Schafer, R. D. Inner derivations of non-associative algebras. Bull. Amer. Math. Soc. **55**, 769–776 (1949).

This paper is concerned with the Lie algebra of derivations of a nonassociative algebra A, i.e., the set $\mathfrak{D}$ of all linear transformations D satisfying the identity $xD \cdot y + x \cdot yD = (xy)D \cdot D$ forms a Lie algebra with respect to the commutator operation $D, D' = DD' - D'D$. The author terms a derivation inner if it is contained in the Lie algebra L generated by all left and right multiplications of A and shows that this coincides with the usual definitions in the case of an associative algebra and also in the case of a Lie algebra. Further, as in these cases, the set of all inner derivations forms an ideal of $\mathfrak{D}$. Finally, he shows that the derivations of semi-simple alternate and Jordan algebras are all inner and in each case the inner derivations take a simple form. Reference is made to the paper by N. Jacobson reviewed above. *D. Rees* (Cambridge, England).

Harish-Chandra. On representations of Lie algebras. Ann. of Math. (2) **50**, 900–915 (1949).

Let $\mathfrak{L}$ be a Lie algebra over a field of characteristic 0; if X_i ($1 \leq i \leq n$) is a base for $\mathfrak{L}$, with the relations $[X_i, X_j] = c^l_{ij} X_l$, form the free algebra $\mathfrak{A}$ with n generators x_i, and consider in $\mathfrak{A}$ the two-sided ideal N generated by the elements $x_i x_j - x_j x_i - c^l_{ij} x_l$; then the algebra $\mathfrak{A}^* = \mathfrak{A}/N$ is called the general enveloping algebra of $\mathfrak{L}$; in this paper, some properties of $\mathfrak{A}^*$ are proved.

Theorem 1: $\mathfrak{A}^*$ has a complete set of finite-dimensional linear representations (the proof is as follows: first, one constructs by Ado's theorem a faithful representation of $\mathfrak{L}$ by matrices of trace 0 in a finite-dimensional vector space V; then it is proved that $\mathfrak{A}^*$ is isomorphic to a subalgebra of the general enveloping algebra $\mathfrak{B}^*$ of the Lie algebra of all matrices of trace zero in V; it remains to prove the theorem for $\mathfrak{B}^*$, which follows from an explicit construction). As a corollary of this theorem it is proved that every Lie algebra has infinitely many nonequivalent irreducible representations.

Theorem 2: any semi-simple Lie algebra over a field of characteristic zero has only a finite number of inequivalent representations whose degree is not greater than r. (The proof uses essentially the well-known results of É. Cartan and H. Weyl concerning the structure of semi-simple Lie algebras).

Theorem 3: let $\mathfrak{A}^*$ be the general enveloping algebra of a semi-simple Lie algebra $\mathfrak{L}$; let $\mathfrak{Z}$ be the center of $\mathfrak{A}^*$; let I_1 and I_2 be the kernels in $\mathfrak{A}^*$ of any two finite-dimensional representations of $\mathfrak{A}^*$; then the relation $I_1 \subset I_2$ holds if and only if $\mathfrak{Z} \cap I_1 \subset \mathfrak{Z} \cap I_2$ holds.

[Reviewer's note. Identify $\mathfrak{L}$ with a subspace of $\mathfrak{A}^*$, and consider in $\mathfrak{A}^*$ the operators $Y \rightarrow [X, Y]$, where $X \epsilon \mathfrak{L}$; it is not difficult to see that this "adjoint representation" is completely reducible, whence follows that, if one denotes by $\mathfrak{A}_0^*$ the subspace of $\mathfrak{A}^*$ generated by the elements $[X, Y]$ ($X, Y \epsilon \mathfrak{A}^*$), $\mathfrak{A}^*$ is the direct sum of $\mathfrak{Z}$ and $\mathfrak{A}_0^*$; consequently, two irreducible finite dimensional representations are equivalent if and only if their characters are equal on the center of $\mathfrak{A}^*$, a result from which theorem 3 follows easily.]

R. Godement (Nancy).

Ado, I. D. The representation of Lie algebras by matrices. Amer. Math. Soc. Translation no. **2**, 21 pp. (1949).

Translated from Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 159–173 (1947); these Rev. 10, 350.

THEORY OF GROUPS

✶Zassenhaus, Hans. The Theory of Groups. Translated from the German by Saul Kravetz. Chelsea Publishing Company, New York, N. Y., 1949. viii+159 pp. $3.50.

The German original was published by Teubner, Leipzig-Berlin, 1937. A supplementary bibliography has been added.

Tuan, Hsio-Fu. An Anzahl theorem of Kulakoff's type for p-groups. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 182–189 (1948).

Let G be a group of order p^n, p an odd prime, whose largest cyclic subgroup has index p^a, so that α is the rank of G in the sense of Hua [same Rep. 4, 313–327 (1947); these Rev. 10, 8]. The number $N(m)$ of subgroups of G of order p^m is determined mod p^3, provided $2\alpha + 1 \leq m \leq n$. If $\alpha = 0$, $N(m) = 1$ for all m, and if $\alpha = 1$, $N(m) = 1 + p$ for $1 \leq m \leq n+1$, so we may assume $\alpha \geq 2$. If the number d of elements in a minimal base of G is greater than 2, theorems of P. Hall [Proc. London Math. Soc. (2) 36, 29–95 (1933)] and Hua [loc. cit.] yield that $N(n-1) = 1+p+p^2$ and $N(m) = 1+p+2p^2$ if $2\alpha + 1 \leq m \leq n-2$. The case $d = 2$, however, requires deeper considerations. The result here is that $N(n-1) = 1+p$; while for $2\alpha+1 \leq m \leq n-2$, $N(m) = 1+p+p^2$, or $N(m) = 1+p+2p^2$, according as a minimal base of the unique subgroup of order p^m and rank α contains two or more than two elements. *G. Higman.*

Makar, Ragy H. The irreducible representations of the symmetric groups of degrees 3, 4 and 5. Proc. Math. Phys. Soc. Egypt 3, 13–21 (1948).

Using the method of W. Specht [Math. Z. 39, 696–711 (1935)] and a result of A. C. Aitken [Proc. Edinburgh

Math. Soc. (2) **7**, 196–203 (1946); these Rev. **8**, 310], the author tabulates integral irreducible representations of the symmetric groups S_3, S_4 and S_5. The reviewer has checked throughout the representations for S_4, which are Young's natural representations. Those for S_5 are not the natural representations but must be equivalent to them. A few random trials revealed no errors.   *D. E. Rutherford.*

✱Peremans, Wouter. Eindige Binaire Projectieve Groepen [Finite Binary Projective Groups]. Thesis, University of Amsterdam, 1949. 54 pp.

F. Klein [Math. Ann. **9**, 183–203 (1876)] has found all the finite subgroups of the group of unitary 2×2 matrices over the complex number field. This thesis considers the group of 2×2 matrices over any commutative field. If the field is modular a parabolic matrix (that is, one whose characteristic roots are equal) may have a finite period. This is the chief difference between this and the complex-number case, for it is shown that the finite subgroups which do not contain parabolic matrices are just those found by Klein. The finite subgroups containing parabolic matrices are also found. In the nonparabolic case, any two of the subgroups which are isomorphic as abstract groups are equivalent as representations. This is not so for the parabolic case. Conditions on the characteristic of the ground field that the theoretically possible subgroups should exist (over a suitably extended ground field) are found. Finally, the conditions for the existence of the subgroups without extension of the ground field are given. These are more complicated, involving, for example, the existence of roots of unity, and for certain of the subgroups the field has to be a splitting field of the quaternion algebra. There is a summary in English by the author.   *H. A. Thurston* (Bristol).

Mal'cev, A. I. On infinite soluble groups. Doklady Akad. Nauk SSSR (N.S.) **67**, 23–25 (1949). (Russian)

Infinite soluble groups, i.e., groups which possess normal series of finite length with Abelian factor-groups, have been studied by several authors: under assumption of the maximal condition by the reviewer [Proc. London Math. Soc. (2) **44**, 53–60, 336–344 (1938); **49**, 184–194 (1946); these Rev. **8**, 132]; under assumption of the minimal condition by S. N. Černikov [e.g., same Doklady (N.S.) **63**, 11–14 (1948); **65**, 21–24 (1949); these Rev. **10**, 590; where also references to earlier papers can be found] and by O. Schmidt [Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 145–162 (1945); these Rev. **7**, 511]. In this paper many earlier results are extended and supplemented. The author classifies soluble groups according to the nature of their Abelian factors. An Abelian group is A_1, if the factor-group with respect to its periodic part has finite rank; A_2, if moreover the periodic part has finite rank, i.e., is the direct product of a finite number of cyclic and locally cyclic subgroups; A_3, if moreover the periodic part is the direct product of a finite number of cyclic and primary locally finite subgroups; A_4, if moreover the periodic part is finite; A_5, if the group has a finite number of generators. A soluble group with Abelian factor-groups of type A_i in a normal series is an A_i-group (A_5-groups are the "S-groups" investigated by the reviewer). Subgroups and direct products of A_i-groups are again A_i-groups. So are factor-groups for $i=1, 2, 5$; for $i=3, 4$ one can only assert this for factor-groups of periodic normal subgroups. The factor-group of an A_1-group with respect to its maximal periodic normal subgroup is an A_4-group.

The author derives his results mainly from the following theorem. A soluble group of matrices with elements in an algebraically closed field contains a subgroup of finite index whose matrices are simultaneously reducible to triangular form. From this he deduces the following results: every soluble A_2-group contains a subgroup of finite index whose commutator-group is nilpotent. If all Abelian subgroups of a locally nilpotent group G are of type A_1, then the factor-group of G with respect to its maximal periodic normal subgroup is a nilpotent A_4-group. A locally nilpotent A_4-group is nilpotent. If all Abelian subgroups of a group G are of type A_3, then G possesses a maximal nilpotent normal subgroup. In particular every A_3-group possesses a maximal nilpotent normal subgroup, and in this case the factor-group is a finite extension of an Abelian subgroup of it. If all soluble subgroups of a group G are A_3-groups, then G possesses a maximal soluble normal subgroup. Further results are as follows. If all Abelian subgroups of a soluble group have type A_5, then the group is an A_5-group. This is based on the lemma: every soluble subgroup of the group of automorphisms of an Abelian group with a finite number of generators is an A_5-group. The periodic subgroups of an A_4-group split into a finite number of conjugate classes.
                K. A. Hirsch (Newcastle-upon-Tyne).

Sato, Shoji. On groups and the lattices of subgroups. Osaka Math. J. **1**, 135–149 (1949).

This is a thorough investigation of the structure of groups G with the following two properties: (1) If A and B are subgroups of G such that A covers $A\cap B$ [i.e., there is no subgroup properly between $A\cap B$ and A], then the compositum $\{A, B\}$ covers B (G is upper semimodular). (2) If U is a normal subgroup of the subgroup V of G, and if the lattice of subgroups of V/U has finite dimension, then V/U is a finite group. Denote by $E=E(G)$ the totality of elements of finite order in G. Then E is a characteristic subgroup of G; and if $G\neq E$, then E is Abelian. If $G=E$, then G is the direct product of indecomposable groups G_τ such that elements in different components G_τ have relatively prime order, and such that G_τ is either a p-group and hence modular or else G_τ is a cyclic extension of a special type of an Abelian group whose elements have square-free order. If U, V are cyclic subgroups of G/E, then either $\{U, V\}$ is cyclic too or else $U\cap V=1$; and if G/E does not contain a perfect subgroup, then G/E is Abelian. If G/E is Abelian of rank 1, then any two subgroups of G commute and G is modular with a rather special structure; and if $G\neq E$ and G does not contain perfect subgroups, then G is modular. This investigation is based on and a continuation of work done by K. Iwasawa [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **4**, 171–199 (1941); these Rev. **3**, 193; and two papers in Japanese].   *R. Baer* (Urbana, Ill.).

Baer, Reinhold. Free sums of groups and their generalizations. An analysis of the associative law. Amer. J. Math. **71**, 706–742 (1949).

A system A is called an add relative to a composition called addition (written $a+b$) if for each a and b in A there exists at most one element c in A with $a+b=c$. In the first part of the paper the author develops a general method for attacking the problem of imbedding adds in semigroups (here called additive manifolds). The investigation centers in various ways for formulating the associative law of addition. Associated with each add A is an additive manifold $D(A)$ called the derived additive manifold of A which is characterized (to within isomorphism) by the properties (i) there is a homomorphism $\mathfrak{r}:x\to x^*=xt$ of A into $D(A)$, (ii) $D(A)$ is generated by the images x^* of elements x in A,

(iii) if $\mathfrak{h}$ is a homomorphism of A into an additive manifold M then there exists a unique homomorphism $\mathfrak{k}$ of $D(A)$ into M such that $x^*\mathfrak{k}=x\mathfrak{h}$ for all x in A. From this it follows that A can be imbedded into an additive manifold if and only if the mapping $\mathfrak{r}$ is an isomorphism. The condition that $\mathfrak{r}$ is an isomorphism is called the third associative law. The weaker condition that $\mathfrak{r}$ is one-to-one is called the second associative law. An ordered set $v=(a_1, \cdots, a_n)$ of elements of A is called a vector. The set of all sums obtained by inserting parentheses in the indicated sum $a_1+\cdots+a_n$ so as to reduce each operation to a single addition is called the summation $S(v)$ of v. The first associative law states that, for any vector v, $S(v)$ consists of at most one element. Although the first associative law is weaker than the second, they are equivalent in the cases of greatest interest. If $v=(a_1, \cdots, a_n)$ and $u=(b_1, \cdots, b_m)$ the sum $w=v+u$ is defined to be the vector $(a_1, \cdots, a_n, b_1, \cdots, b_m)$. If r and s are vectors and if $a+b=c$ then the vectors $r+a+b+s$ and $r+c+s$ are said to be strictly similar to each other. Two vectors u and v are said to be similar (written $u\sim v$) if there exists a sequence $u=u_1, u_2, \cdots, u_k=v$ in which each adjacent pair are strictly similar. The third associative law states that $u\sim v$ only if $S(u)=S(v)$. The fourth law is stronger than the third. Various conditions are given under which equivalences between these laws hold. The general theory of adds is then applied to the problem of generation of groups by subadds with attention to questions of freeness and independence, and to the problem of determination of free sums of groups with an amalgamated subgroup. New proofs are given of certain of the results of O. Schreier [Abh. Math. Sem. Hamburg. Univ. 5, 161–183 (1927)] and of H. Neumann [same J. 70, 590–625 (1948); these Rev. 10, 233]. *R. M. Thrall* (Ann Arbor, Mich.).

Følner, Erling. A proof of the main theorem for almost periodic functions in an Abelian group. Ann. of Math. (2) 50, 559–569 (1949).

In a previous note [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 356–362; these Rev. 8, 368] the author has described the adaptation of a sharp lemma of Bogolioùboff from the straight line to any Abelian group, the purpose being to obtain a proof for the completeness of characters from approximation by countable groups. In the present paper the author supplies the proofs that were omitted in his first note, and he also notes that the method gives a very direct proof for the fact that any sequence of Bochner-Fejér polynomials, as formed with characters, if formally convergent is also uniformly so. *S. Bochner.*

Riss, Jean. Transformation de Fourier des distributions. C. R. Acad. Sci. Paris 229, 12–14 (1949).

Let S be the space of all functions on the locally compact Abelian group G which are continuous, continuously differentiable to all orders, and integrable after any derivation and multiplication by a polynomial, in the sense of previous definitions of the author [C. R. Acad. Sci. Paris 227, 664–666, 809–810, 1194–1195 (1948); these Rev. 10, 429], and whose Fourier transforms have the same properties. Theorem 1 asserts that every $f\varepsilon S$ is the limit, for every $a\varepsilon G$, in a certain locally convex topology on S, of a sequence which is the sum of a sequence of finite linear combinations of characters and a sequence of functions vanishing outside neighborhoods of a. Theorem 2 states that every S-distribution (i.e., continuous linear functional on S) whose support is a single point a, is a finite linear combination of

derivatives of the translation through a of the Dirac function. *I. E. Segal* (Chicago, Ill.).

Fukamiya, Masanori. Topological method for Tauberian theorem. Tôhoku Math. J. (2) 1, 77–87 (1949).

It is shown that if I is a closed ideal in the L_1-algebra of a locally compact Abelian group G, with the property that for every continuous character x^* on G there is an element in I whose Fourier transform does not vanish at x^*, then $I=L_1(G)$. This is a known generalization of the first half of Wiener's general Tauberian theorem [cf. Segal, Trans. Amer. Math. Soc. 61, 69–105 (1947); these Rev. 8, 438], but the paper was submitted for publication in 1942. *I. E. Segal* (Chicago, Ill.).

Michal, Aristotle D. Differentiable infinite continuous groups in abstract spaces. Revista Ci., Lima 50, 131–140 (1948).

This paper is, in the author's words, "a preliminary account of some applications of the differential calculus in abstract spaces to the fundamentals of the theory of infinite continuous groups." The development is founded on a group G whose elements form an open set in a Banach space. The group operations are assumed to possess continuous Fréchet differentials of orders one or two for most of the work. The paper sketches rapidly such topics as the Lie total differential equations of the parameter groups, the structural form, and the generalized Maurer-Cartan equations. *A. E. Taylor* (Los Angeles, Calif.).

Gleason, A. M. A note on locally compact groups. Bull. Amer. Math. Soc. 55, 744–745 (1949).

Proof of the following theorem: a locally compact group G is isomorphic to a closed subgroup of a unimodular locally compact group whose space is $G\times R$. Let dx be the left-invariant Haar measure in G, and put $d(xa)=\Delta(a)\cdot dx$; then the theorem is proved by introducing in $G\times R$ the multiplication defined by $(g, r)(g', r')=(gg', r\cdot\Delta(g')^{-1}+r')$. *R. Godement* (Nancy).

Wang, Hsien-Chung. Homogeneous spaces with non-vanishing Euler characteristic. Acad. Sinica Science Record 2, 215–219 (1949).

A homogeneous space W is a space on which a topological group G acts transitively. If G is a connected compact Lie group and W has a nonvanishing Euler characteristic then W is called a χ space. If G is simple then W is called an elementary χ space, and it is shown that every χ space is homeomorphic to a topological product of elementary χ spaces. If O_r is a real proper orthogonal matrix of order r, then all the matrices O_{2i} form a group denoted by D_r^0. The author here limits himself to χ spaces associated with these groups and calls them χ_D spaces. He remarks that similar methods can be used to study homogeneous spaces of all the four classes of simple groups. The group D_r^0 contains a subgroup isomorphic to the unitary group whose matrices are denoted by M_{2i}. Suppose a set $\Omega=\{e; a_1, \cdots, a_n; d_1, \cdots, d_m\}$ of integers to be given such that $e-n+\sum a_i+\sum d_i=l$, $e\geq n$, $a_i\geq 2$, $d_i\geq 2$, $n\geq 0$, $m\geq 0$. Consider the matrices in D_l^0 consisting of the following blocks along the main diagonal:

$$O_2^1, \cdots, O_2^{e-n}, O_{2a_1}, \cdots, O_{2a_n}, M_{2d_1}, \cdots, M_{2d_m}.$$

These matrices form a subgroup L of D_l^0 and $W_D^0=D^0/L$ is an elementary χ space. The space W_D^0 is simply connected and every elementary χ_D space is covered by a W_D^0. The

Euler characteristic is given by a formula and the first five homotopy groups are also determined explicitly. The details will appear elsewhere.   *D. Montgomery.*

Samelson, Hans. Sur les sous-groupes de dimension 3 des groupes de Lie compacts. C. R. Acad. Sci. Paris **228**, 630–631 (1949).

The author gives a simple proof of the fact that a non-Abelian three-dimensional subgroup Q of a compact Lie group G is not homologous to 0 in G: the invariant differential form of degree 3 of G (explicitly constructed in terms of the constants of structure) induces on Q the invariant differential form of degree 3 of Q, which is not equal to 0.
                    C. Chevalley (New York, N. Y.).

Leray, Jean. Détermination, dans les cas non exceptionnels, de l'anneau de cohomologie de l'espace homogène quotient d'un groupe de Lie compact par un sous-groupe de même rang. C. R. Acad. Sci. Paris **228**, 1902–1904 (1949).

Let X be a connected compact Lie group, product of simple groups belonging to the four classes. Let Y be a subgroup of X, T an Abelian subgroup of Y. Assume that X, Y, T are of equal rank l. Let N_T be the normalisor of T in X. Let Φ_Y be the subgroup $(Y \cap N_T)/T$ of the finite group $\Phi_X = N_T/T$. The group Φ_X operates on T and on the homogeneous space X/T. Let $H(X), \cdots$ denote the cohomology ring of $X, \cdots$ over a fixed commutative field of characteristic zero. The theorem states in part that (a) the natural mapping of X/T onto X/Y has as reciprocal an isomorphism of $H(X/Y)$ into $H(X/T)$, the image coinciding with the totality of elements of $H(X/T)$ invariant under Φ_Y; (b) if Y is connected, and if the Poincaré polynomials of $H(X)$, $H(Y)$ are $\prod(1+s^{p_\lambda})$, $\prod(1+s^{q_\lambda})$, $p_\lambda = 2m_\lambda - 1$, $q_\lambda = 2m_\lambda - 1$, $\lambda = 1, \cdots, l$, that of X/Y is

$$\prod(1+s^{p_\lambda+1})/\prod(1+s^{q_\lambda+1});$$

(c) if Z is the identity component of Y, the fundamental group of X/Y is Y/Z which is isomorphic with Φ_Y/Φ_Z and operates on X/Z and $H(X/Z)$ in a specified manner. The author uses technical devices introduced in various earlier notes [notably same vol., 1545–1547, 1784–1786 (1949); these Rev. 10, 680; 11, 10].   *P. A. Smith.*

Dynkin, E. B. On the representation by means of commutators of the series log $(e^x e^y)$ for noncommutative x and y. Mat. Sbornik N.S. **25(67)**, 155–162 (1949). (Russian)

The Campbell-Hausdorff theorem asserts that the polynomials P_n, occurring in the expansion log $(e^x e^y) = \sum P_n(x, y)$, can be expressed in terms of commutators. In a preceding paper [Doklady Akad. Nauk SSSR (N.S.) **57**, 323–326 (1947); these Rev. **9**, 132] the author derived a canonical form for any polynomial expressible in terms of commutators. This applies in particular to P_n, assuming that the Campbell-Hausdorff result is known. [The latter was not proved in the cited paper, the statement in the review to that effect being in error.] In the present paper a new proof is given of the Campbell-Hausdorff theorem, and the proof also yields a new formula for P_n.   *I. Kaplansky.*

Tamari, Dov. Ordres pondérés. Caractérisation de l'ordre naturel comme l'ordre du semi-groupe multiplicatif des nombres naturels. C. R. Acad. Sci. Paris **229**, 98–100 (1949).

This note is a continuation of an earlier one [same C. R. **228**, 1909–1911 (1949); these Rev. **11**, 9]. Let S be an Archimedean, totally (=simply) ordered semi-group, the order relation Ω_r in S being "double and conservative," i.e., $a > b$ implies $ac > bc$ and $ca > cb$. Then Ω_r is a "weighted order of S": the weight $L_p(q)$ of q with respect to p is a finite real number for every $p, q \varepsilon S$. Among the results announced [without proof] are the following. For a fixed $p \varepsilon S$, the mapping $q \to L_p(q)$ is an order-preserving homomorphism of S into the additive group R of real numbers. If this mapping is one-to-one, Ω_r is called "perfect." If S possesses a perfect ordering, it must be Abelian. An example of an imperfect order is afforded by the ordering Ω_l of the free semi-group on more than one generator, since any two words of the same length have the same weight. [Note by the reviewer: Ω_r is perfect, for Abelian S, if and only if it remains Archimedean when extended to the quotient group G of S.] If S is the free Abelian semi-group on a set A of generators, then Ω_r is perfect and discrete if and only if the set $\pi(A)$ of weights of elements of A is discrete and linearly independent. If S is the multiplicative semi-group of the natural numbers, and G the multiplicative group of the rationals, then $L_p(q) = \log_p q$ $(p, q \varepsilon G)$. But for any ordering of S other than the natural one, $L_p(q)$ is a discontinuous solution of the functional equation $L(qr) = L(q) + L(r)$.
                    A. H. Clifford (Baltimore, Md.).

NUMBER THEORY

Tietze, Heinrich. Tafel der Primzahl-Zwillinge unter 300000. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 57–72 (1949).

The table gives the larger twin $p+2$ of the twin primes $p, p+2$ for each such case up to 300,000. There are 2994 pairs of twin primes below this limit. The primes were taken originally from a list by Kraitchik [Recherches sur la théorie des nombres, Gauthier-Villars, Paris, 1924, table I] and a complete list of errata is also published in this note.

                    D. H. Lehmer (Berkeley, Calif.).

Jarden, Dov, and Katz, Alexander. Table of binary linear recurring sequences of order 3. Riveon Lematematika **2**, 54–55 (1948). (Hebrew)

The authors consider four third-order recurring series defined by

$$\begin{aligned}
U_n &= U_{n-2} + U_{n-3}; & U_0 &= 0, \ U_1 = 0, \ U_2 = 1; \\
V_n &= V_{n-2} + V_{n-3}; & V_0 &= 3, \ V_1 = 0, \ V_2 = 2; \\
\bar{U}_n &= -\bar{U}_{n-2} + \bar{U}_{n-3}; & \bar{U}_0 &= 0, \ \bar{U}_1 = 0, \ \bar{U}_2 = 1; \\
\bar{V}_n &= -\bar{V}_{n-2} + \bar{V}_{n-3}; & \bar{V}_0 &= 3, \ \bar{V}_1 = 0, \ \bar{V}_2 = -2.
\end{aligned}$$

These are tabulated for $-64 \leq n \leq 64$. The factorizations of these values are given completely with the exception of $V_{62} = 2 \times 9803919989$ where the large factor may be composite.   *D. H. Lehmer* (Berkeley, Calif.).

✦Kaprekar, D. R. Demlo Numbers. Published by the author, Khareswada, Devlali, India, 1948. x+114 pp. 6 Rupees.

A Demlo number is a number in the scale of ten formed by writing consecutively the digits of a number M, a digit r repeated any number of times, and a number P, where the decimal expansion of $M+P$ contains only the digit r (any number of times); the case $M=P=0$, $r\neq0$ is also allowed. This volume contains 20 notes on the subject, of which 16 have been published in journals.

Inkeri, K. Some extensions of criteria concerning singular integers in cyclotomic fields. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 49, 15 pp. (1948).

Let l be an odd prime and write $\zeta=\exp(2i/l)$. An integer in the cyclotomic field $K(\zeta)$ is said to be singular if it is equal to the lth power of a nonprincipal ideal in K. Necessary conditions for an integer to be singular have been given by Fueter, Hasse, Takagi and Vandiver. In the present paper these criteria are extended to the case in which l is replaced by $m=l^r$. Criteria of Kummer and Furtwängler for the equation $x^l+y^l+z^l=0$ are extended to cover the case of $x^m+y^m+z^m=0$. *D. H. Lehmer* (Berkeley, Calif.).

Kapferer, Heinrich. Über ein Kriterium zur Fermatschen Vermutung. Comment. Math. Helv. **23**, 64–75 (1949).

The following theorem is proved: the Fermat equation $x^p+y^p+z^p=0$ (p a prime greater than 7) has no integral solution with the restriction

$$xyz(x-y)(y-z)(z-x)(x^2+y^2+z^2)\not\equiv0 \pmod{p}$$

provided the discriminant of the polynomial

$$\sum_{(r)}\binom{\frac12(p-3)-r}{2r}(2r+1)^{-1}t^r$$

is not divisible by p. This criterion is proved in an entirely elementary manner by deriving a necessary and sufficient condition that the congruence $x^p+y^p+z^p\equiv0 \pmod{p^2}$ have a solution with the restriction on x, y, z mentioned above; this condition comes out in terms of the discriminant of the theorem. The author notes that, for $p<100$, the discriminant in question is divisible by p only for $p=59$, 79, 83. *H. W. Brinkmann* (Swarthmore, Pa.).

Izvekoff, J. Sur une propriété des nombres premiers. Bull. Soc. Math. Phys. Serbie **1**, 41–43 (1949). (Serbian. Russian and French summaries)

The author shows that there are no solutions of Fermat's equation, $a^n=b^n+c^n$, $n>2$; a, b, c rational integers, for which a is a prime. [Cf. Abel, Oeuvres, v. 2, 1881, p. 254.] *R. Bellman* (Stanford University, Calif.)

Perez-Cacho, L. The function which is the sum of Euler functions of successive orders. Memorias de Matemática del Instituto "Jorge Juan," no. 7, 25 pp. (1948). (Spanish)

Let $\varphi_1(n)=\varphi(n)$ denote Euler's function and let $\varphi_r(n)=\varphi(\varphi_{r-1}(n))$ be its rth iteration. The author sets $P(n)=\sum_{i=1}^m\varphi_i(n)$ where m is the smallest index for which $\varphi_m(n)=1$, and proves various properties of this function. Thus he proves that $P(n)=2n-3$ if and only if n is a Fermat prime $2^{2^k}+1$ and that $P(n)=n-1$ if and only if n is of the form $n=2^k$. In a previous paper [Revista Mat. Hisp.-Amer. (3) **1**, 45–50 (1939); these Rev. **1**, 290] the author listed various cases where the relation $P(n)=n$ holds.

Another simple result states that, if $m=P(n)$, then N exists so that $2m+1=P(N)$. The author conjectures that every even integer is expressible in the form $P(n)+P(m)$, but believes that the proof would be of the same order of difficulty as a proof of the Goldbach conjecture. *H. W. Brinkmann* (Swarthmore, Pa.).

Storchi, E. Congruenze che caratterizzano i numeri primi. Period. Mat. (4) **27**, 117–121 (1949).

It is shown that the congruence

$$[1!2!3!\cdots(p-1)!]^2\equiv(-1)^{(p+1)/2} \pmod{p}$$

is a necessary and sufficient condition that p be a prime. The left hand side is the square of the Vandermonde determinant formed with the first p integers. *R. Bellman.*

Longo, Carmelo. Sui residui n^{iei} d'un modulo primo p. Atti Accad. Ligure **2**, 83–86 (1942).

By actual calculation of the resultant of the two equations $x^n-a=0$, $x^{N-1}-1=0$, the author derives the known necessary and sufficient condition that a be an nth residue modulo a prime. *R. Bellman.*

Gyires, B. Über die Faktorisation im Restklassenring mod m. Publ. Math. Debrecen **1**, 51–55 (1949).

Let $N_r(a, m)$ denote the number of solutions of the congruence $a\equiv x_1x_2\cdots x_{r+1} \pmod{m}$, two solutions $x_i=b_i$ and $x_i=c_i$ being considered identical if and only if $b_i\equiv c_i \pmod{m}$, $i=1, 2, \cdots, r+1$. The author shows that

$$N_r(a, m)=N_r((a, m), m),$$

and that $N_r(d'd'', m'm'')=N_r(d', m')N_r(d'', m'')$ provided $d'|m'$, $d''|m''$, $(m', m'')=1$. The determination of $N_r(a, m)$ is completed by proving that, for a prime p,

$$N_r(p^t, p^e)=\binom{t+r}{t}\varphi(p^e), \qquad 0\leq t<e,$$

$$N_r(p^e, p^e)=\sum_{k=0}^r\binom{k+e-1}{k}p^{e(r-k)}\varphi^k(p^e),$$

where φ is the Euler function. A further result is that

$$\sum_{a=1}^m N_r(a, m)=\sum_{d|m}\varphi(m/d)N_r(d, m)=m^{r+1}.$$

The paper concludes with a discussion of the behavior of $N_r(a, m)$ for large r. *N. H. McCoy.*

Podsypanin, V. D. On the indeterminate equation $x^3=y^2+Az^6$. Mat. Sbornik N.S. **24(66)**, 391–403 (1949). (Russian)

Points (x, y) on $C(A):x^3-A=y^2$ can be written as $x=\wp(t)$, $y=\frac12\wp'(t)$, where $\wp$ is the Weierstrass elliptic function of invariants $g_2=0$, $g_3=4A$. Let $A=f^2g\neq0$ be an integer, f^2 its greatest square factor, hence g squarefree. By Mordell's theorem [Proc. Cambridge Philos. Soc. **21**, 179–192 (1922)], the arguments t of the points $T(x, y)$ on $C(A)$ with rational x, y form an Abelian group G of finite basis; denote by $a_1T_1+a_2T_2$ the point of argument $a_1t_1+a_2t_2$ if T_1 and T_2 belong to t_1 and t_2, respectively, and a_1, a_2 are integers. By a theorem of Fueter, there corresponds to every rational point $T(x, y)$ on $C(A)$ a rational point (x', y') on $C(-27A)$ defined by

$$(1) \qquad x'=\frac{x^3-4A}{x_2}, \qquad y'=\frac{y(x^3+8A)}{x^3},$$

while, conversely, there corresponds to this point (x', y') a

new rational point $T^*(x^*, y^*)$ on $C(A)$ given by

$$(2) \qquad x^* = \frac{x'^2 + 108A}{z'^2}, \quad y^* = \frac{y'(x'^2 - 216A)}{z'^3}.$$

On eliminating x', y' from (1) and (2), it is seen that $T^* = 3T$. Denote by H the set of all rational points T^* on $C(A)$ derivable from a rational point (x', y') on $C(-27A)$, but not necessarily of the form $3T$ where $T \epsilon G$; $T^*(x^*, y^*)$ belongs to H if and only if $y^* + \sqrt{(-A)}$ is the cube of an element of the quadratic field $R(\sqrt{(-A)}) = R(\sqrt{(-g)})$. One shows that H is a subgroup of G; evidently $G \supset H \supset 3G$, where $3G$ consists of all points $3T$ with $T \epsilon G$ and is a subgroup of H. Both quotient groups G/H and $H/3G$ are direct products of cyclical groups of order 3, say of k and l such groups, respectively. Since on changing over from $C(A)$ to $C(-27A)$ these two numbers k and l are interchanged, it suffices to investigate k. The author gives for this number the upper bound $k \leq n_1 + n_2 + n_3 + n_4$, where n_1 is the number of prime factors of f which split in $R(\sqrt{(-g)})$ into prime ideals of the first order prime to the discriminant of this field; 3^{n_2} is the number of ideal classes of $R(\sqrt{(-g)})$ the cube of which is the principal class; n_3 is 1 when $g \equiv 7 \pmod 8$ and 0 otherwise; and n_4 is 0 when $g > 0$ and $g \neq 3$, and 1 otherwise. The paper concludes with a table of the bases of G for $1 \leq |A| \leq 89$ and the values of k and l in these cases.

<div align="right">K. Mahler (Manchester).</div>

Buquet, A. Comparaison de différentes solutions de l'équation diophantienne (1) $x^2 + y^2 + z^2 = t^2$. Mathesis **58**, 70–73 (1949).

Zahlen, Jean-Pierre. Sur l'équation Diophantienne $X^3 + Y^3 + Z^3 = hT^3$. Euclides, Madrid **9**, 139–142 (1949).

Infinitely many solutions X, Y, Z, T of the indicated equation are given in terms of 3 independent parameters and h. I. Niven (Eugene, Ore.).

Gloden, A. Résolution d'un système diophantien. Euclides, Madrid **9**, 218–219 (1949).

Five different sets of 2-parameter families of solutions are given for the simultaneous solution of the Diophantine system $A^4 + B^4 + C^4 = D^4 + E^4 + F^4$, $A^2 + B^2 = D^2$, $C^2 = E^2 + F^2$. [Equation (2) should be $A^2 + B^2 = D^2$ and the fourth subsequent equation should be $D = \rho(a^2 + b^2)$.] I. Niven.

Gloden, A. Zwei Parameterlösungen einer mehrgradigen Gleichung. Arch. Math. **1**, 480–482 (1949).

By application of theorem 2 [Amer. Math. Monthly **53**, 205–206 (1946)] with the additional condition $a_3 - a_1 = b_2 - b_1$ or $a_2 - a_1 = b_3 - b_2$ two-parameter solutions of

$$\sum_{i=1}^{5} x_i^n = \sum_{i=1}^{5} y_i^n,$$

$n = 1, 3, 5, 7$, are deduced. N. G. W. H. Beeger.

Gloden, A. Über mehrgradige Gleichungen. Arch. Math. **1**, 482–483 (1949).

Solution of $\sum_{i=1}^{5} x_i^n = \sum_{i=1}^{5} y_i^n$, $n = 1, 2, \cdots, 5$, and of $\sum_{i=1}^{4} x_i^n = \sum_{i=1}^{4} y_i^n$, $n = 1, 3, 5$. N. G. W. H. Beeger.

Hua, Loo-Keng. Improvement of a result of Wright. J. London Math. Soc. **24**, 157–159 (1949).

The result referred to is in a paper by E. M. Wright [same J. **23**, 279–285 (1948); these Rev. **10**, 510] on a problem of sets of integers having equal sums of like powers.

The functions $L(k, s)$ and $W(k, s)$ used below are defined there. Wright has shown that if $H(k, s)$ is the greatest integer not exceeding

$$\log \{\tfrac{1}{2}(k+1)(s+1)+k^{-1}-k)\}/\log (1+k^{-1}) \sim k \log k$$

then $L(k, s) \leq (k+1)H(k, 2)$ for all s and

$$W(k, s) \leq (k+1)H(k, s)$$

but was unable to find an upper bound for $W(k, s)$ independent of s, as in the case of $L(k, s)$. The present paper proves that Wright's result for $L(k, s)$ holds also for $W(k, s)$ although $L(k, s) \leq W(k, s)$. D. H. Lehmer.

Gupta, Hansraj. A solution of the Tarry-Escott problem of degree r. Proc. Nat. Inst. Sci. India **15**, 37–39 (1949).

The Tarry-Escott problem of order q and of degree precisely r is that of finding q sets $S_1, S_2, \cdots, S_q$ of s integers each such that the members of each set have equal sums of kth powers for $k = 0, 1, \cdots, r$ but distinct sums for $k = r + 1$. The least s for which this problem is possible depends on r and possibly also on q and is denoted by $M_q(r)$. It is known that $M_q(r) = r + 1$ for $r = 1, 2, 3, 5$. The author proves that $M_q(r+1) \leq q M_q(r)$ so that $M_q(4) \leq 4q$, $M_q(r) \leq 6q^{r-5}$ $(r \geq 5)$. [For a much stronger inequality see the paper by E. M. Wright cited in the preceding review.] It is conjectured that $M_q(r)$ is not a function of q. [An upper bound for $M_q(r)$ independent of q has been obtained by Hua in the paper reviewed above.] D. H. Lehmer (Berkeley, Calif.).

Mordell, L. J. Rational points on cubic surfaces. Publ. Math. Debrecen **1**, 1–6 (1949).

Lecture at the University of Debrecen.

Obrechkoff, Nikola. Sur l'approximation diophantique linéaire. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 283–285 (1949).

By a modification of the Schubfachprinzip the following sharp form of Dirichlet's theorem is proved. Let $\omega_1, \cdots, \omega_m$ be arbitrary real numbers and let n be a positive integer. Then there exist integers $x_1, \cdots, x_m$ not all zero and an integer y such that $|x_i| \leq n$ for $i = 1, \cdots, n$ and

$$\left| \sum_{i=1}^{m} \omega_i x_i - y \right| \leq (n+1)^{-m};$$

the equality sign holds if and only if the numbers ω_i are all rational and have, in some order, the denominators $n+1$, $(n+1)^2$, $\cdots$, $(n+1)^m$. The author also shows that, if $a_1, \cdots, a_n$ are integers, a necessary and sufficient condition that the form $\sum_{i=1}^{m} a_i x_i$ represent every integer (mod n^m) when the x_i vary independently over a complete residue system (mod n) is that the a's, in some order, are numbers $b_i n^i$, $i = 0, \cdots, m-1$, where the numbers b_i are all prime to n. W. J. LeVeque (Ann Arbor, Mich.).

Kubilyus, I. On the application of I. M. Vinogradov's method to the solution of a problem of the metric theory of numbers. Doklady Akad. Nauk SSSR (N.S.) **67**, 783–786 (1949). (Russian)

Let $s \geq 1$ be a rational integer and c any positive constant. Then Mahler showed [Math. Ann. **106**, 131–139 (1932)] that, for almost all real numbers θ and for $\omega \geq 3$ the inequality

$$\left| \sum_{v=0}^{s} a_v \theta^v \right| < ca^{-(1+\omega)s}, \qquad a = \max_{0 \leq v \leq s} |a_v|,$$

possesses only a finite number of solutions in rational integers $a_1, a_2, \cdots, a_s$. Koksma replaced the condition $\omega \geq 3$ by

$\omega \geqq 2$ [Monatsh. Math. Phys. **48**, 176–189 (1939); these Rev. **1**, 137] and it is conjectured that the result holds for any positive ω. That this is so for $s=1$ is almost trivial, and the purpose of the author's paper is to prove the conjecture for $s=2$. His proof is based on an estimate of Vinogradov for a trigonometric sum and upon the following lemma. Let $\varphi(q)$ be a positive decreasing function of the integer q for $q \geqq q_0 > 0$ such that the series $\sum_{q \geqq q_0} q^1 \varphi(q) \tau(q) \log^1 q$ is convergent, where $\tau(q)$ is the number of divisors of q. Then the system of inequalities $|\theta - p_1/q| < \varphi(q)$, $|\theta^2 - p_2/q| < \varphi(q)$ possesses only a finite number of solutions for almost all θ. The main result then follows from this lemma by taking a particular function $\varphi(q)$ and applying a known result due to Khintchine and Mahler [Mahler, Rec. Math. [Mat. Sbornik] N.S. **1**(43), 961–962 (1936)]. *R. A. Rankin.*

Korobov, N. M. **On sums of fractional parts.** Doklady Akad. Nauk SSSR (N.S.) **67**, 781–782 (1949). (Russian)

Let q be an integer greater than unity and let $\{\beta\}$ denote the fractional part of β. Further, let A be the set of numbers $\alpha \varepsilon (0, 1)$ for which the function $\{\alpha q^x\}$ is equally distributed. Then the following results are stated. (I) If $\epsilon(P)$ is any positive function of P which tends to zero as $P \to \infty$, then there exists at least one $\alpha \varepsilon A$ such that $\sum_{x=1}^{P} \{\alpha q^x\} - \frac{1}{2} P = \Omega(P\epsilon(P))$. (II) If $\varphi(P)$ is any increasing function of P which tends to infinity with P as slowly as we please, then there exists at least one $\alpha \varepsilon A$ such that $\sum_{x=1}^{P} \{\alpha q^x\} - \frac{1}{2} P = o(\varphi(P))$. Proofs are not given. It is stated that (I) is obtained from Weyl's conditions for equal distribution [Math. Ann. **77**, 313–352 (1916)] and that (II) and a further result which is not precisely stated, are based on applications of ideas introduced in a paper of the author which appears to be unpublished. *R. A. Rankin* (Cambridge, England).

Gel'fond, A. O. **On the algebraic independence of transcendental numbers of certain classes.** Doklady Akad. Nauk SSSR (N.S.) **67**, 13–14 (1949). (Russian)

Three general theorems are stated which, it is claimed, can be deduced by a variation of the methods used in an earlier paper [same Doklady (N.S.) **64**, 277–280 (1949); these Rev. **10**, 682]. These theorems are too complicated to state here, but the following consequences may be mentioned. (I) If a and α are algebraic numbers and $a \neq 0, 1$, and if the degree of α exceeds 2, then it is not possible to express each of the four numbers a^α, a^{α^2}, a^{α^3}, a^{α^4} algebraically in terms of one of them. When α is a cubic irrational it follows that a^α and a^{α^2} are algebraically independent in the field of the rational numbers, a result already proved in the paper referred to. (II) If a is as before and ν is rational and not zero, then it is not possible to express each of the four numbers a^ν, $a^{2\nu}$, $a^{3\nu}$, $a^{4\nu}$ algebraically in terms of one of them and, in particular, at least one of them is transcendental. A similar result holds for the first three of the four numbers. *R. A. Rankin* (Cambridge, England).

Varnavides, P. **Quadratic forms near to $x^2 - 2y^2$.** Quart. J. Math., Oxford Ser. **20**, 124–128 (1949).

Heinhold [Math. Z. **44**, 659–688 (1939)] zeigte, dass die Ungleichung $|f(x+x_0, y+y_0)| \leqq \frac{1}{2} = \frac{1}{4}(d/2)^{\frac{1}{2}}$ für

$$f(x, y) = f_0(x, y) = x^2 - 2y^2$$

und irgendwelche reellen Zahlen x_0, y_0 in ganzzahligen x, y stets lösbar ist (d bedeutet die Diskriminante der quadratischen Form $f(x, y)$) [Nederl. Akad. Wetensch., Proc. **51**, 396–404 = Indagationes Math. **10**, 142–150 (1948); diese Rev. **10**, 19] bewies eine Verschärfung dieser Ungleichung

auf $\frac{1}{4} = \frac{1}{2}(d/2)^{\frac{1}{2}}$, falls nur $x_0 = 0(1)$, $y = \frac{1}{2}(1)$ nicht gilt. Nun wird gezeigt, dass eine solche Verschärfung auch möglich ist für Formen $f(x, y)$, die "nahe bei $f_0(x, y)$" liegen. Der Satz lautet: seien g und h beliebige reelle Zahlen mit $7/5 < g < 10/7$, $7/5 < h < 10/7$ und sei $f(x, y) = (x - gy)(x + hy)$ mit $d = (g+h)^2$. Dann existieren für beliebige reelle x_0, y_0 ganze Zahlen x, y, dass $|f(x+x_0, y+y_0)| < d^{\frac{1}{2}}/5.898 < 0.96 \cdot \frac{1}{2}(d/2)^{\frac{1}{2}}$ gilt, ausser wenn $g = h = \sqrt{2}$ und $x_0 = 0(1)$, $y = \frac{1}{2}(1)$ erfüllt sind.
T. Schneider (Göttingen).

Rogers, C. A. **On the critical determinant of a certain nonconvex cylinder.** Quart. J. Math., Oxford Ser. **20**, 45–47 (1949).

If C is a plane star body of points (x, y), let K be the 3-dimensional cylindrical star body of all points (x, y, z), $|z| \leqq 1$. Let $\Delta(S)$ denote the lower bound of all determinants of lattices with no interior points of the star body S other than the origin. The author constructs a nonconvex star body C, bounded by straight lines, with $\Delta(C) < \Delta(K)$. This is in contrast to the earlier result that $\Delta(C) = \Delta(K)$ if C is convex [J. H. H. Chalk and C. A. Rogers, J. London Math. Soc. **23**, 178–187 (1948); these Rev. **10**, 284].
D. Derry (Vancouver, B. C.).

Chalk, J. H. H., and **Rogers, C. A.** **Corrigendum: The critical determinant of a convex cylinder.** J. London Math. Soc. **24**, 240 (1949).

The paper appeared in the same J. **23**, 178–187 (1948); these Rev. **10**, 284.

✱**Hinčin, A. Ya.** **Tri Žemčužiny Teorii Čisel. [Three Pearls of the Theory of Numbers].** OGIZ, Moscow-Leningrad, 1947. 72 pp.

The three chapters are entitled: van der Waerden's theorem on arithmetic progressions; the Landau-Schnirelmann hypothesis and Mann's theorem; elementary solution of Waring's problem [Linnik, Rec. Math. [Mat. Sbornik] N.S. **12**(54), 225–230 (1943); these Rev. **5**, 200].

Mirsky, L. **A theorem on representations of integers in the scale of r.** Scripta Math. **15**, 11–12 (1949).

Let $\alpha^{(r)}(n)$ denote the sum of the digits of the representation of n in the scale of r. The author proves easily

$$\sum_{n=1}^{x} \alpha^{(r)}(n) = \frac{1}{2}(r-1)x \log x / \log r + O(x)$$

and shows that this result is best possible.
H. Heilbronn (Bristol).

Mirsky, L. **On the distribution of integers having a prescribed number of divisors.** Simon Stevin **26**, 168–175 (1949).

Denote by $\Delta(x, k)$ the number of positive integers not exceeding x and having exactly k divisors. An asymptotic expression is obtained for $\Delta(x, k)$ for fixed k and as $x \to \infty$. In the proof of the result use is made of the prime number theorem and known estimates for the sums $\sum_{p \leqq x} p^{-1}$ and $\sum_{p \leqq x} p^{-1} \log p$. An asymptotic formula is also given for the number of positive integers not exceeding x and having at most k divisors. *W. H. Simons* (Vancouver, B. C.).

LeVeque, Wm. J. **On the size of certain number-theoretic functions.** Trans. Amer. Math. Soc. **66**, 440–463 (1949).

The major results of this paper are the following. (I) Let $d(m)$ and $\nu(m)$ denote the number of divisors of m and the number of distinct prime divisors of m, respectively; let ω

be a real number. The number of positive integers $m \leq n$ for which $d(m) < 2^{\epsilon(n)}$, $\epsilon(n) = \log_2 n + \omega(\log_2 n)^{\frac{1}{2}}$, or for which $\nu(m) < \epsilon(n)$, is $nD(\omega) + O(n \log_3 n/(\log_2 n)^{\frac{1}{2}})$, where

$$D(\omega) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\omega} e^{-x^2/2} dx.$$

(II) Let $f(m)$ be a strongly additive function, that is, $f(mn) = f(m) + f(n)$ if $(m, n) = 1$ and $f(p^a) = f(p)$. Suppose that $|f(p)| \leq 1$ for all primes p. Let $\sum_p f^2(p)/p = \infty$, and let ω_1, ω_2 be any real numbers. Then the number of positive integers $m \leq n$ for which simultaneously $f(m) < A_n + \omega_1 B_n$ and $f(m+1) < A_n + \omega_2 B_n$, where

$$A_n = \sum_{p \leq n} f(p)/p, \quad B_n^2 = \sum_{p \leq n} f^2(p)/p,$$

is $nD(\omega_1)D(\omega_2) + o(n)$. This implies a result stated without proof by Erdős [Ann. of Math. (2) **47**, 1–20 (1946); these Rev. **7**, 416]. (III) The number of $m \leq n$ for which $d(m) < 2^{f(n)}d(m+1)$, $f(n) = \omega(2 \log_2 n)^{\frac{1}{2}}$, or for which $\nu(m) < \nu(m+1) + f(n)$ is $nD(\omega) + o(n)$. These theorems are extensions in various directions of results obtained by Erdős and Kac [Amer. J. Math. **62**, 738–742 (1940); these Rev. **2**, 42] and are derived by similar methods. The principal tools are the central limit theorem from the theory of probability and the Viggo Brun sieve method. Improvements in error terms are obtained by using a hitherto unpublished theorem of J. B. Rosser and W. J. Harrington which states a complicated result obtained by the use of Brun's method.   *A. L. Whiteman* (Los Angeles, Calif.).

Erdős, P. Problems and results on the differences of consecutive primes. Publ. Math. Debrecen **1**, 33–37 (1949).

Let $m = \epsilon \log n$, where n is a large integer and ϵ is a small but fixed number; let $f(m)$ be a function which tends to infinity with m and such that $f(m) = o((\log m)^{\frac{1}{2}})$; let N be the product of the primes not exceeding m. Employing the method of Chang [Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin **4**, 33–55 (1938)], the author demonstrates the existence of a residue class $x \pmod{N}$ such that $(x+1, N) = 1$ and $(x+k, N) \neq 1$ for all k for which $|k| \leq mf(m)$, $k \neq 1$. Making use of the theorem of Linnik [Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 139–178, 347–368 (1944); these Rev. **6**, 260] on the least prime in an arithmetic progression, and a classical inequality of prime number theory, he deduces the following theorem. Let $d_n = p_{n+1} - p_n$ be the difference between two consecutive primes. Then

$$\lim \sup \, (\min \, (d_n, d_{n+1})/\log n) = \infty;$$

that is, corresponding to every positive constant c, there exist values of n satisfying the inequalities $d_n > c \log n$, $d_{n+1} > c \log n$. A novel feature of the argument is that it does not employ the Brun method.   *A. L. Whiteman.*

Buhštab, A. A. On those numbers in an arithmetic progression all prime factors of which are small in order of magnitude. Doklady Akad. Nauk SSSR (N.S.) **67**, 5–8 (1949). (Russian)

The author proves the following theorem. Suppose $l < k$, $(l, k) = 1$; denote by $B_l(k, x, y)$ the number of positive integers in the progression $kn + l$ not exceeding x and free of prime factors greater than y; then for $\alpha \geq 1$ we have $B_l(k, x, x^{1/\alpha}) = x\omega(\alpha)/k + O(x\{\log x\}^{-1})$, where $\omega(\alpha)$ is a positive continuous monotonically decreasing function of α

given for $N \leq \alpha \leq N+1$ by

$$\omega(\alpha) = 1 + \sum_{n=1}^{N} (-1)^n \int_n^\alpha \int_{n-1}^{n-1} \cdots$$
$$\times \int_1^{z_{n-1}-1} (z_1 \cdots z_n)^{-1} dz_1 \cdots dz_n.$$

The proof is by induction on the positive integer N. The author seems to be unaware that the case $k=1$ was treated earlier by Chowla and Vijayaraghavan [J. Indian Math. Soc. (N.S.) **11**, 31–37 (1947); these Rev. **9**, 332]; the generalization to any k is straightforward.

It is not difficult [cf. Ramaswami, Duke Math. J. **16**, 99–109 (1949), lemma 2(a); these Rev. **10**, 597] to prove that $\alpha^{-1\alpha} < \omega(\alpha) < 1/\Gamma(\alpha+1)$ for $\alpha > 1$. Using the integro-difference equation

$$\omega(\alpha) - \omega(\beta) = \int_\alpha^\beta z^{-1} \omega(z-1) dz$$

satisfied by $\omega(\alpha)$, the author improves the first of these inequalities by proving that for $\alpha \geq 6$ we have

$$(*) \quad \omega(\alpha) > \exp(-\alpha\{\log \alpha + \log \log \alpha + 6(\log \log \alpha)/(\log \alpha)\}).$$

[N. G. de Bruijn has recently obtained still sharper results which are as yet unpublished.]

Finally the author points out that, if m is an integer not less than 2, p is a prime, $m | (p-1)$, and $\omega(\alpha) > 1/m$, then the least mth-power nonresidue modulo p is less than $p^{1/(2\alpha)}$ for sufficiently large p. Using this remark and (*) the author proves that for $m \geq 2^{33}$ the least mth-power nonresidue modulo p is less than $p^{(\log \log m+2)/(2 \log m)}$ for sufficiently large p; this improves a result of Vinogradov [Trans. Amer. Math. Soc. **29**, 218–226 (1927), theorem 4] by roughly a factor 2 in the exponent.   *P. T. Bateman* (Princeton, N. J.).

Min, Szu-Hoa. On the order of $\zeta(1/2 + it)$. Trans. Amer. Math. Soc. **65**, 448–472 (1949).

The author proves that $\zeta(\frac{1}{2} + ti) = O(|t|^\theta)$ for $\theta > 15/92$. Previously the best proved result was

$$\zeta(\frac{1}{2} + ti) = O(|t|^{19/116} \log^{1/58} |t|)$$

[Titchmarsh, Quart. J. Math., Oxford Ser. **13**, 11–17 (1942); these Rev. **4**, 131]. The author's proof is a refinement of Titchmarsh's proof, the details of which are too complicated to be explained shortly.   *H. Heilbronn.*

Chowla, S., and Selberg, A. On Epstein's zeta function. I. Proc. Nat. Acad. Sci. U. S. A. **35**, 371–374 (1949).

Without detailed proofs the following results are announced. (1) $\sum_{n=1}^{\infty} (-p/n)n^{-s} > 0$ for $s > 0$, $p = 43, 67$ or 163. (2) Let K and K' have the usual meaning in the theory of elliptic functions and let $d < 0$ be the discriminant of a quadratic field of class-number h, containing w roots of unity. Then if iK/K' belongs to the field, the number

$$K\pi^{-1} \left\{ \prod_{m=1}^{|d|-1} \Gamma(m/|d|)^{(d/m)} \right\}^{-\frac{1}{2}w/h}$$

is algebraic. (3) The function $\sum'(x^2 + y^2 + ds^2)^{-s}$ has a real zero which tends to 0 as $d \to \infty$.   *H. Heilbronn.*

Rodosskiĭ, K. A. On the zeros of Dirichlet's L-functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 315–328 (1949). (Russian)

The author proves that if $\log \log D \leq \Psi(D) \leq \frac{1}{2} \log D$, then the number of L-series formed with characters mod D

which have at least one zero in the rectangle

$$1 - \Psi(D)/\log (D(|T|+2)) \leqq \sigma \leqq 1,$$
$$|t - T| \leqq K \log^2 (D(|T|+2))$$

does not exceed $Be^{A\Psi(D)} \log^5 (D(|T|+2))$, where A, B, K are absolute constants. This is a slight improvement on a previous result by the author [same Izvestiya Ser. Mat. 12, 47–56 (1948); these Rev. 9, 413]. From this he derives the theorem about the distribution of primes in an arithmetical progression in the following form (using the conventional notation):

$$\varphi(D)\psi(x; D, l) = x - E\chi(\rho)x^\beta \beta^{-1} + xO(\exp (-A_3 \log x/\log D))$$

uniformly in D and l for $A_1 \log D \log \log D \leqq \log x \leqq A_2 \log^2 D$. Here β is the largest positive zero of all the $L(s, \chi)$, χ the corresponding character mod D, and $E=1$ or 0 if such a zero exists or does not exist. This again is a slight improvement on a previous result by the author [same Izvestiya Ser. Mat. 12, 123–128 (1948); these Rev. 9, 499].

H. Heilbronn (Bristol).

Jacobinski, Heinz. Über die Automorphismen einer quadratischen Form. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 19, no. 8, 17 pp. (1949).

The author extends two theorems to an arbitrary field of characteristic not 2. The first is due to Frobenius [J. Reine Angew. Math. 85, 185–213 (1878)] for an algebraically closed field and the other is due to Loewy [Pascal, Repertorium der höheren Mathematik, v. 1, 2d ed., Teubner, Leipzig, 1910, in particular, p. 136] for a real closed field.

L. K. Hua (Peking).

Sominskiĭ, I. S. On the existence of automorphs of the second order for certain indefinite quadratic forms. Amer. Math. Soc. Translation no. 1, 25 pp. (1949). Translated from Mat. Sbornik N.S. 23(65), 279–296 (1948); these Rev. 10, 357.

van der Blij, F. On the theory of quadratic forms. Ann. of Math. (2) 50, 875–883 (1949).

Let S and T be symmetric positive matrices with integer elements and of orders m and n, respectively, with $m \geqq n$. Let P be an m by n integral matrix the greatest common divisor of whose n-rowed minors is 1 and $A(S, T; P, \nu)$ the number of solutions of $X'SX = T$ with $X \equiv P(\text{mod } \nu)$, while

a subscript q on A indicates that the equation is replaced by the congruence $X'SX \equiv T(\text{mod } q)$. The principal class $PC(S, \nu)$ of S consists of all matrices $U'SU$, where U is a unimodular matrix for which $U \equiv E(\text{mod } \nu)$. The matrix S_1 belongs to the principal genus $PG(S, \nu)$ of S if, for every q, there is a matrix U satisfying the conditions ($|U|, q) = 1$, $U'SU \equiv S_1(\text{mod } q)$, $U \equiv E(\text{mod } \nu)$. For $\nu > 2$, by a result of Minkowski pointed out by C. L. Siegel, $M(S, \nu)$ reduces to the number of principal classes in the principal genus of S. Define $M(S, T; P, \nu) = \sum A(S_k, T; P, \nu)$, the sum being over all principal classes S_k in the principal genus of S and $A_0(S, T; P, \nu) = M(S, T; P, \nu)/M(S, \nu)$. The author proves

$$A_0(S, T; P, \nu) = \epsilon \delta, \gamma_{mn} |S|^{-\frac{1}{2}n} |T|^{\frac{1}{2}(m-n-1)}$$
$$\times \lim_{q \to \infty} A_q(S, T; P, \nu)/(\alpha q^{mn - \frac{1}{2}n(n+1)}),$$

where, if $m \neq n$, $\alpha = \delta, = 1$ while if $m = n$, $\alpha = 2^{\omega(q)}$; $\omega(q)$ is the number of prime factors of q, ϵ and γ_{mn} are explicitly given in terms of m and n and δ, in terms of ν. A few consequences are noted.

B. W. Jones (Boulder, Colo.).

✶Landau, Edmund. Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. Chelsea Publishing Company, New York, N. Y., 1949. vii+147 pp. $2.95.

Photographic reprint of the second edition, published by Teubner, Leipzig, 1927.

Kurbatov, V. A. On the monodromy group of an algebraic function. Mat. Sbornik N.S. 25(67), 51–94 (1949). (Russian)

A polynomial $F(x)$ with integral coefficients is said to produce a substitution, for a prime p, if, for every pair of integers x_1 and x_2 which are incongruent modulo p, $F(x_1)$ and $F(x_2)$ are incongruent. This notion is due to Dickson. A positive integer n was termed by I. Schur a Dickson number if every $F(x)$ of degree n which produces a substitution for infinitely many primes can be derived by linear transformations from x^n or from the Chebyshev polynomial of degree n. Schur proved that every odd prime is a Dickson number. In this paper, it is proved that products of three or of four distinct odd primes, and products of powers of two odd primes, are Dickson numbers. The proofs, which are intricate, employ the group of monodromy of the inverse of $F(x)$ and the notion of imprimitivity. J. F. Ritt.

ANALYSIS

Varopoulos, Th. Sur quelques propriétés des polygones convèxes. Prakt. Akad. Athēnōn 14, 424–427 (1939). (French. Greek summary)

Let $\varphi(x)$ be a convex function of x on the interval $I: 0 \leqq x \leqq s$. At each point $X = (x_1, \cdots, x_n)$ of the $n-1$ simplex S consisting of those points X with x_r in I and $\sum x_r = s$ define the function $\sigma(X) = \sum \varphi(x_r)$. It is shown that σ attains its minimum value at the centroid $\bar{X}$ with coordinates $\bar{x}_r = s/n$ of S. A proof of this, shorter than the one given by the author, is:

$$\sigma(\bar{X}) = n\varphi(s/n) = n\varphi(\sum x_r/n) \leqq n \sum \varphi(x_r)/n = \sigma(X),$$

the inequality being strict if φ is strictly convex and $X \neq \bar{X}$. In application the x_r are taken to be the angles of a convex n-gon. W. Gustin (Bloomington, Ind.).

Fempl, S. Sur la limite supériéure de la différence de l'intégrale d'un produit et du produit des intégrales. Bull. Soc. Math. Phys. Serbie 1, 21–27 (1949). (Serbian. Russian and French summaries)

The author gives a simplified proof of the mean-value theorem

$$(b-a)^{-1} \int_a^b f(t)g(t)dt - (b-a)^{-2} \int_a^b f(t)dt \int_a^b g(t)dt$$
$$= \frac{1}{4}\{f(t_1) - f(t_2)\}\{g(t_1) - g(t_2)\},$$

where t_1 and t_2 are in $[a, b]$. This theorem was proved by Karamata and implies several older inequalities [Karamata, Acad. Serbe Sci. Publ. Inst. Math. 2, 131–145 (1948); these Rev. 10, 435]. The author also gives a geometrical interpretation. R. P. Boas, Jr. (Providence, R. I.).

Tomić, Miodrag. **Théorème de Gauss relatif au centre de gravité et son application.** Bull. Soc. Math. Phys. Serbie 1, 31–40 (1949). (Serbian. Russian and French summaries)

The theorem of Gauss is that the center of gravity of n mass-points is in the smallest closed circumscribed convex polygon. The author uses it to prove Jensen's inequality and some more general inequalities.   *R. P. Boas, Jr.*

Bilimovitch, Anton D. **On the maximum values of determinant's modul.** Glas Srpske Akad. Nauka 189, 195–199 (1946). (Serbian. English summary)

The Hadamard theorem for the maximum value of a determinant is sharpened. Let $\Delta = |a_{ij}|$ be an nth order determinant, and let $p_{ij} = \sum_{s=1}^{n} a_{is} a_{js}$, $p_i^2 = p_{ii}$, $i, j = 1, 2, \cdots, n$. A sequence of determinants D_k is obtained such that $1 = D_1 \geqq D_2 \geqq \cdots \geqq D_n$, and $(p_1 p_2 \cdots p_n)^2 D_n = \Delta^2$. Then each D_k, $k = 2, 3, \cdots, n$, gives an improvement on Hadamard's original bound $\Delta^2 \leqq (p_1 p_2 \cdots p_n)^2$. In particular, $D_2 = 1 - p_{12}^2/p_1^2 p_2^2$ and
$$D_3 = 1 - p_{23}^2/(p_2^2 p_3^2) - p_{31}^2/(p_1^2 p_3^2) - p_{12}^2/(p_1^2 p_2^2) + 2 p_{23} p_{31} p_{12}/(p_1^2 p_2^2 p_3^2).$$
The proof is essentially that given by Hadamard [Bull. Sci. Math. (2) 17, 240–246 (1893)] modified to include terms that were originally rejected.   *A. W. Goodman.*

Wilkins, J. Ernest, Jr. **A bound for the mean value of a function.** Bull. Amer. Math. Soc. 55, 801–803 (1949).

Let $f(t)$ be measurable for $0 \leqq t \leqq \pi$, and let $|f(t)| \leqq 1$. If $\int_0^\pi f(t) \sin(2n+1) t \, dt = 0$, then
$$\left| \int_0^\pi f(t) dt \right| \leqq \{4(n+1) \sec^{-1}(2n+2) - \pi\}/(2n+1).$$
The inequality is the best possible.   *P. Civin.*

Kolmogoroff, A. **On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval.** Amer. Math. Soc. Translation no. 4, 19 pp. (1949).

Translated from Učenye Zapiski Moskov. Gos. Univ. Matematika 30, 3–16 (1939); these Rev. 1, 298.

Schoenberg, Isaac J., et Whitney, Anne. **Sur la positivité des déterminants de translation des fonctions de fréquence de Pólya, avec une application à un problème d'interpolation.** C. R. Acad. Sci. Paris 228, 1996–1998 (1949).

A Pólya frequency function $\Lambda(x)$ is characterized by the nonnegativeness of the determinants $D = \det \|\Lambda(x_i - y_j)\|_{1,n}$ for all n and all choices of x_i, y_j with $x_i < x_{i+1}$, $y_j < y_{j+1}$. As shown by Schoenberg [Proc. Nat. Acad. Sci. U. S. A. 33, 11–17 (1947); these Rev. 8, 319], $\Lambda(x)$ has a bilateral Laplace transform $[\Phi(s)]^{-1}$ where
$$\Phi(s) = e^{-\gamma s^2 - \delta s} \prod_1^\infty (1 + \delta_\nu s) e^{\delta_\nu s}, \quad \gamma \geqq 0, \quad 0 < \gamma + \sum_1^\infty \delta_\nu^2 < \infty.$$
The object of the present note is to determine when D is actually positive. The authors show that $D > 0$ if $\gamma > 0$ or if $\gamma = 0$ and $\sum |\delta_\nu| = \infty$. If $\sum |\delta_\nu| < \infty$, they give necessary and sufficient conditions on the x_i, y_j which make $D > 0$. These conditions are trivially satisfied if there are infinitely many sign changes in the sequence $\{\delta_\nu\}$. The conditions are applied to the case where $\Lambda(x) = x^k$ for $0 \leqq x$ and 0 for $x < 0$; the corresponding determinants play a role in the theory of Dirichlet series. The conditions are also applied for deciding when an interpolation problem $F(x_i) = Y_i$,

$i = 1, 2, \cdots, n+k+1$, has a unique solution. Here $F(x)$ is a given spline function, i.e., $F(x) = P_{k,j}(x)$, a polynomial of degree not exceeding k in the interval $I_j = (\xi_j, \xi_{j+1})$, $\xi_0 = -\infty$, $\xi_{n+1} = +\infty$, so chosen that $F(x)$ is continuous together with its first $k-1$ derivatives.   *E. Hille* (New Haven, Conn.).

Laasonen, Pentti. **Einige Sätze über Tschebyscheffsche Funktionensysteme.** Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 52, 24 pp. (1949).

The functions $\varphi_i(x)$ $(0 \leqq i \leqq n)$ form a T-system in (a, b) if no linear combination $\varphi(x)$ of the $\varphi_i(x)$ with coefficients not all zero can have more than n zeros in (a, b). Let $\psi(x)$ be an arbitrary function such that $\varphi_i(x)$ $(0 \leqq i \leqq n)$ and $\psi(x)$ form a T-system. If $\varphi(x)$ is the linear combination minimizing (*) $\int_a^b |\psi(x) - \varphi(x)| dx$ then the difference $\psi(x) - \varphi(x)$ has at most $n+1$ zeros x_ν and these zeros are independent of $\psi(x)$. Thus to a given $\psi(x)$ satisfying the condition mentioned above the integral (*) will be a minimum if $\psi(x)$ coincides with $\varphi(x)$ at the points x_ν. In the special case $\varphi_i(x) = x^i$, $-1 \leqq x \leqq 1$, we have $x_\nu = \cos \nu\pi/(n+2)$, $\nu = 1, 2, \cdots, n+1$. As an application the following is proved. Let $|f^{(n+1)}(x)| \geqq m > 0$ in $-1 < x < 1$. Then for arbitrary c_ν,
$$\int_{-1}^{1} |f(x) \sum_{\nu=0}^{n} c_\nu x^\nu| dx > 2^{-n} m/(n+1)!.$$
Finally the construction of T-systems is discussed.   *G. Szegö* (Stanford University, Calif.).

Popken, J. **A property of a Dirichlet series, representing a function satisfying an algebraic difference-differential equation.** Nederl. Akad. Wetensch., Proc. 52, 499–504 = Indagationes Math. 11, 159–164 (1949).

Unter Verwendung von Gedanken einer Arbeit von Ostrowski [Math. Z. 8, 241–298 (1920)] wird eine neue notwendige Bedingung für Dirichletsche Reihen, die einer algebraischen Differenzen-Differentialgleichung der Form
$$F(s, f^{(n_1)}(s+u_1), \cdots, f^{(n_t)}(s+u_t)) = 0$$
genügen, bewiesen, wobei $F(x_0, x_1, \cdots, x_t) \not\equiv 0$ ein Polynom bedeutet, $u_1, \cdots, u_t$ reelle Zahlen und $n_1, \cdots, n_t$ nichtnegative ganze Zahlen, dass die t Systeme $(u_1, n_1), \cdots, (u_t, n_t)$ voneinander verschieden sind. Die Bedingung lautet: Die konvergente Dirichletsche Reihe $\sum_{\nu=1}^{\infty} a_\nu e^{-\lambda_\nu s}$ mit nichtverschwindenden Koeffizienten a_ν stelle eine Funktion $f(s)$ dar, die einer algebraischen Differenzen-Differentialgleichung genügt. Dann gibt es eine positive Zahl c derart, dass $\lambda_h > h^c$ für alle h, ausser einer endlichen Anzahl, gilt. Daraus folgt sofort speziell, dass $\zeta(s)$ transzendental-transzendent ist.   *T. Schneider* (Göttingen).

Calculus

★**Bourbaki, N.** **Éléments de mathématique. IX. Première partie: Les structures fondamentales de l'analyse. Livre IV: Fonctions d'une variable réelle (théorie élémentaire). Chapitre I: Dérivées. Chapitre II: Primitives et intégrales. Chapitre III: Fonctions élémentaires.** Actualités Sci. Ind., no. 1074. Hermann et Cie., Paris, 1949. ii+184 pp.

This installment of Bourbaki's super-textbook gives a notable account of Rolle's theorem and Taylor's theorem with remainder; of the indefinite integral, as anti-derivative, for a function having only discontinuities of the first kind,

such a function being a uniform limit of a function which is constant by intervals, an "interval" being an open interval or a point; of Cauchy limits for such integrals; of integration and differentiation with respect to a parameter under the integral sign; and of the elementary logarithmic, exponential and trigonometric functions. Everything is done for abstractly-valued functions whose values lie in a topologized vector-space, or, at the very least, in a normed ring, but there are many fine features of analysis spread out underneath this superimposed layer of ever-present generalizations.

Over and above this there is a thirty-five page chapter on the history of the calculus from Archimedes to C. Jordan and it is a very vivacious account with plenty of information and sidelights; and it again and again broaches the unanswerable question of why Archimedes did not succeed in discovering the calculus all by himself, seeing that he all but had it and was quite capable of doing so. All the talk about Greek mathematics having been fated to bloom in geometry is not very convincing, and the mystery of the missing calculus is darker than ever.   *S. Bochner.*

✶v. Mangoldt, H. **Einführung in die höhere Mathematik für Studierende und zum Selbststudium. Vollständig neu bearbeitet und erweitert von Konrad Knopp. Band I, Zahlen, Funktionen, Grenzwerte, Analytische Geometrie, Algebra, Mengenlehre; Band II, Differentialrechnung, Unendliche Reihen, Elemente der Differentialgeometrie und der Funktionentheorie; Band III, Integralrechnung und ihre Anwendungen, Funktionentheorie, Differentialgleichungen.** 9th ed. S. Hirzel Verlag, Stuttgart, 1948. xv+585+xv+634+xvi+618 pp. DM 24.80 per volume.

Only slight modifications have been made since the 6th edition [vol. 1, 1931; vol. 2, 1932; vol. 3, 1933]. The 7th and 8th editions of vol. 1 appeared in 1942 and 1944, respectively; of vol. 2, in 1942 and 1947; of vol. 3, in 1942 and 1944.

✶Hildebrand, F. B. **Advanced Calculus for Engineers.** Prentice-Hall, Inc., New York, N. Y., 1949. xiii+594 pp.

Solutions of linear ordinary differential equations; The Laplace transformation; Numerical methods for solving ordinary differential equations; Series solutions of differential equations; Boundary-value problems and orthogonal functions; Vector analysis; Partial differential equations; Solutions of partial differential equations of mathematical physics; Functions of a complex variable.
                                          Table of contents.

Puig Adam, Pedro. **A general theorem on integrals of composite functions and its geometrical and physical applications.** Revista Mat. Hisp.-Amer. (4) **9**, 16–25 (1949). (Spanish)

The contents of this paper are well-known elementary results of "Duhamel type" for Riemann integrals. Evidently the author was unaware of such papers on this topic as those of W. F. Osgood [Ann. of Math. (2) **4**, 161–178 (1903)], R. L. Moore [ibid. **13**, 161–166 (1912)], and G. A. Bliss [ibid. **16**, 45–49 (1914)]; in particular, the author's approach is the same as that of Bliss.   *W. T. Reid.*

Kašanin, Radivoje. **L'introduction en arithmétique de l'angle, des fonctions trigonométriques et du nombre π.** Glas Srpske Akad. Nauka **191**, 149–161 (1948). (Serbian. French summary)

The author claims that the theory of functions borrows the number π and the trigonometric functions from geom-

etry. He shows that the functional equations for sin and cos follow from the binomial expansion $(a\pm bi)^n = A(n)\pm iB(n)$.
                                          W. Feller (Ithaca, N. Y.).

Witt, Ernst. **Rekursionsformel für Volumina sphärischer Polyeder.** Arch. Math. 1, 317–318 (1949).

Let H_ν $(\nu=1, \cdots, n)$ denote n hemispheres on an N-dimensional sphere of volume 1. Let $H_\alpha H_\beta \cdots H_\gamma$ denote the volume of the intersection of $H_\alpha, \cdots, H_\gamma$. Then the following formula is obtained:

$$H_1 \cdots H_n = 2 \sum_{\substack{k\ \text{even} \\ n < \cdots < n-k+1}} (2^k-1)k^{-1}B_k H_{n1} \cdots H_{n-k+1},$$

where n is odd and B_k is the kth Bernoulli number (in the even-suffix notation).   *L. Carlitz* (Durham, N. C.).

Goodstein, R. L. **On the evaluation of Planck's integral.** Edinburgh Math. Notes 37, 17–20 (1949).

Démonstration élémentaire de la formule

$$\int_0^\infty (e^x-1)^{-1}x\,dx = \tfrac{1}{6}\pi^2.$$
                                          A. Ghizzetti (Pisa).

Ostrowski, A. M. **On some generalizations of the Cauchy-Frullani integral.** Proc. Nat. Acad. Sci. U. S. A. 35, 612–616 (1949).

The classic Frullani formula

$$(1) \qquad \int_0^\infty t^{-1}\{f(at)-f(bt)\}dt = [f(\infty)-f(0)]\log(a/b),$$

in which a and b are positive constants, is generalized in two ways. If $f(t)$ is Lebesgue integrable over each interval $0 < h \le t \le H < \infty$ and the mean values

$$(2) \qquad M(f)=\lim_{x\to\infty} x^{-1}\int_0^x f(t)dt, \quad m(f)=\lim_{\epsilon\to0+}\int_\epsilon^1 t^{-2}f(t)dt$$

exist, then the Frullani integral in the left member of (1) exists and has the value (3) $[M(f)-m(f)]\log(a/b)$. Moreover, if the integral in (1) exists for pairs of positive values of a and b such that the numbers $\log(a/b)$ fill a set having positive measure, than the mean values in (2) exist.

The second generalization replaces at and bt by more general functions. When $g(t)$, $\varphi(t)$ and $\psi(t)$ satisfy appropriate conditions, the integral

$$(4) \qquad \int_0^\infty [g(\psi(t))\psi'(t) - g(\varphi(t))\varphi'(t)]dt$$

can clearly be attacked by methods applicable to Frullani's integral, and

$$(5) \qquad M(xg(x))\log\lim_{x\to\infty}\{\psi(x)/\varphi(x)\}$$
$$-m(xg(x))\log\lim_{x\to0}\{\psi(x)/\varphi(x)\}$$

is given as the canonical form of the value of the integral.
                                          R. P. Agnew (Ithaca, N. Y.).

Bremekamp, H. **On Euler's sine product.** Simon Stevin 26, 203–213 (1949). (Dutch)

Expository lecture, giving several well-known proofs for the sine product, and a new one, suggested by a "proof" given by Euler [Introductio in Analysin Infinitorum, Lausanne, 1748, vol. 1, chapter 9], and depending on the identity

$$(1+x/n)^n - (1-x/n)^n = 2x \prod_{k<\frac{1}{2}n} \{1+x^2/n^2 \tan^2(k\pi/n)\}.$$

                                          J. Korevaar (Lafayette, Ind.).

Verblunsky, S. On Green's formula. J. London Math. Soc. 24, 146–148 (1949).

The author gives a brief proof of Green's formula, employing the method used by Estermann [Math. Z. 37, 556–560 (1933); 38, 641 (1934)] to prove the strong form of Cauchy's theorem.   *W. T. Reid* (Evanston, Ill.).

✱**Schönhardt, Erich.** Vektor-Rechnung mit je einem Anhang über Tensoren und über komplexe Zahlen und Zeiger. Edited by Werther Pavel. 2d ed. E. Schönhardt, Stuttgart, 1948. 295 pp. 12 DM.

A course of vector algebra and analysis for technical students. It contains a large number of worked examples.
                                    L. M. Milne-Thomson (Greenwich).

Theory of Sets, Theory of Functions of Real Variables

✱**Borel, Émile.** Éléments de la Théorie des Ensembles. Éditions Albin Michel, Paris, 1949. 319 pp. 720 francs.

This volume, which is designed for the reader with only modest mathematical attainments, contains a lucid and concise exposition of various topics in the theories of infinite sets, topology, probability, and measure, all referred to the real line. Familiarity with the elementary properties of the real number system is tacitly assumed, but no other mathematical background is needed. Chapter I deals with denumerable sets, the basic facts being illustrated by examples and ordinal numbers being defined by reference to successive derived sets. Chapter II, on continuous sets, contains besides the standard facts on cardinal numbers a discussion of closed and perfect subsets of the line, the Cantor set, and Peano's curve. In chapter III the author discusses choice and probability, illustrating his statements with numerous examples, and describing some of his own classical results on denumerable probabilities. Chapter IV is devoted to probabilities in sets of real numbers defined by properties of decimal expansions. The theories of measure developed by Jordan, the author, and Lebesgue are outlined in chapter V, while chapter VI presents a detailed study of sets of Lebesgue measure zero. This chapter contains a number of recent results; the exposition is somewhat marred by the use of symbols which are not satisfactorily defined. In chapter VII the author discusses the axiom of choice and its consequences. As one might expect, there is a strong philosophical tendency in this chapter. The volume is completed by five notes containing computational details omitted in the main text. This book is one which can be read with interest and profit by an intelligent student who has an understanding of the calculus; the more expert will be interested in the author's individual opinions.
                                        E. Hewitt (Seattle, Wash.).

✱**Hausdorff, Felix.** Grundzüge der Mengenlehre. Chelsea Publishing Company, New York, N. Y., 1949. viii+476 pp. $4.95.

This is a photographic reprint of the [first] edition of 1914 [Veit, Leipzig], containing material which was omitted from later editions.

Eggleston, H. G. The fractional dimension of a set defined by decimal properties. Quart. J. Math., Oxford Ser. 20, 31–36 (1949).

Any real number x in (0, 1) being expressed in the scale of N, let $P(x, i, r)$ be the number of times that the digit r occurs in the first i digits of this expression. The paper proves a conjecture of I. J. Good that the set of x for which $P(x, i, r) \sim p_r i$ (as $i \to \infty$, for each r) has fractional dimension α given by $N^{-\alpha} = \prod p_r^{p_r}$.   *H. D. Ursell* (Leeds).

Denjoy, Arnaud. L'introduction d'un nouvel élément dans un ensemble ordonné. C. R. Acad. Sci. Paris 229, 570–573 (1949).

The following two theorems are proved. (I) It is impossible to construct an ordered set E such that the introduction of a new element a into E, no matter where, leaves the order type of E unchanged. (II) It is possible to construct a denumerable ordered set E such that the introduction of a new element a between any two nonvoid segments of E leaves the order type of E unchanged.   *I. L. Novak.*

Dinghas, Alexander. Über einen Satz von Felix Behrend. Math. Nachr. 2, 141–147 (1949).

Let R_n denote a Riemann space of constant curvature K, M a point set in R_n and M_ϵ the set of points of R_n whose distance from M is at most ϵ. The author proves that if $K > 0$ and M is any set, M_ϵ is Jordan-measurable and that the same is true if $K \leq 0$ and M is bounded. This is a generalisation of the corresponding result in Euclidean space due to F. Behrend [Math. Ann. 111, 289–292 (1935)].

The author considers the metric

$$[ds]^2 = \sum_1^{n-1} dy_i^2 + \frac{K(\sum_1^{n-1} y_i dy_i)^2}{1 - K\sum_1^{n-1} y_i^2} + (1 - K\sum_1^{n-1} y_i^2)dy_n^2$$

and shows that it may be reduced to the standard form $[ds]^2 = dx_1^2 + \cdots + dx_n^2 + dx_{n+1}^2/K$ where

$$(1) \qquad x_1^2 + \cdots + x_n^2 + x_{n+1}^2/K = 1/K$$

by an appropriate transformation from the $y_1, \cdots, y_n$ to the $x_1, \cdots, x_{n+1}$. The properties of the transformation are then used to establish geometrical relations between sets of points in the R_n which satisfy (1) and the corresponding sets of points in Euclidean space with coordinates $(y_1, \cdots, y_n)$. These relations lead to several results of which the most important is stated above. [There is a misprint in (4.7) on p. 145: $\psi^2(\delta)$ should be $\phi^2(\delta)$. The succeeding argument is not immediately applicable to the spherical case, $K > 0$, since $\frac{1}{2}(y_n - y_n') \cdot K^{\frac{1}{2}}$ may be near π. When $K \leq 0$ the hypothesis "y_n bounded" is required.]   *H. G. Eggleston.*

Esenin-Vol'pin, A. S. On the relation between the local and integral weight in dyadic bicompacta. Doklady Akad. Nauk SSSR (N.S.) 68, 441–444 (1949). (Russian)

Let γ be any cardinal number, and let D_γ be the Cartesian product of a family of cardinal number γ of T_1-spaces, each containing exactly two points. A Hausdorff space is said to be a dyadic bicompactum if it is the continuous image of some space D_γ. For any cardinal number γ, let $\chi(\gamma)$ denote the least cardinal number such that γ is the sum of $\chi(\gamma)$ cardinal numbers each less than γ. The following theorems are proved. (1) The character of any infinite dyadic bicompactum is the upper bound of all of its point characters. (2) If a dyadic bicompactum X has character m and if $(m) > \aleph_0$, then X contains a point whose character is m.
                                        E. Hewitt (Seattle, Wash.).

Lyapunov, A. A. On effective measurability. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 357–362 (1949). (Russian)

Novikov [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 1939, 35–40] has introduced the notion

of effectively noncountable set and has shown that such a set lying in a Baire (0-dimensional) space contains a perfect subset. In the present paper the author gives analogous definitions of set effectively not of measure zero and of set effectively not of the first category. Let E be a set of irrational numbers between 0 and 1. It is shown that (1) if E is effectively not of measure zero, then E contains a subset of positive Lebesgue measure, and (2) if E is effectively not of the first category, then E contains a G_δ not of the first category. *H. L. Smith* (Baton Rouge, La.).

Kawada, Yukiyosi. **On equivalence of measures on an infinite product space.** Math. Japonicae 1, 170–177 (1949).

The purpose of this paper is to investigate conditions under which two measures defined on the combinatorial product $X^* = \mathfrak{P}_{n=1}^\infty X_n$ of a sequence of sets $\{X_n\}$ are equivalent, i.e., have the same 0-sets. For the case of direct product measures this problem has been solved by Kakutani [Ann. of Math. (2) 49, 214–224 (1948); these Rev. 9, 340]. Given a Borel field $\mathfrak{B}_n$ on each X_n, the following Borel fields of subsets of X^* are formed: (a) $\mathfrak{I}^*$, generated by all sets of the form $\mathfrak{P}_{n=1}^\infty Y_n$, where $Y_n \varepsilon \mathfrak{B}_n$ for each n and $Y_n = X_n$ for all but a finite number of n; (b) $\mathfrak{I}^{(n)*}$, consisting of all elements of $\mathfrak{I}^*$ of the form $A \times \mathfrak{P}_{i=n+1}^\infty X_i$ with $A \subset \mathfrak{P}_{i=1}^n X_i$. Then a result of B. Jessen [Acta Math. 63, 249–323 (1934)] is generalized as follows. Let m and m' be two countably additive measures on $\mathfrak{I}^*$ with $m(X^*) = m'(X^*) = 1$, and let $m^{(n)*}$ and $m'^{(n)*}$ be their restrictions to $\mathfrak{I}^{(n)*}$, respectively. Let $f(x)$ and $f_n(x)$ be the density functions corresponding to the Lebesgue decompositions of m' relative to m and of $m'^{(n)*}$ relative to $m^{(n)*}$, respectively. Then we have $\lim_{n\to\infty} f_n(x) = f(x)$ m-almost everywhere on X^*. Corollary: m and m' are equivalent if and only if $m^{(n)*}$ and $m'^{(n)*}$ are equivalent for each n and $f_n(x)$ converges m-almost everywhere on X^* to a function with m-integral 1. The paper ends with a new proof of the result of Kakutani mentioned above. *H. M. Schaerf* (St. Louis, Mo.).

Stampacchia, Guido. **Sulle successioni di funzioni continue rispetto ad una variabile e misurabili rispetto ad un'altra.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 198–201 (1949).

The author generalizes Egoroff's theorem to a sequence of functions $f_n(x, y)$ $(n = 1, 2, \cdots)$ defined in the rectangle R: $(a \le x \le b, c \le y \le d)$, continuous with respect to y, and measurable with respect to x. Theorem I. Suppose, moreover, that (a) the functions $f_n(x, y)$ are equicontinuous as functions of y for almost all x of $[a, b]$, and that (b) for every y of $[c, d]$ the sequence $\{f_n(x, y)\}$ converges almost everywhere in $[a, b]$. Then to every $\epsilon > 0$ one can determine a measurable set s of $[a, b]$ with measure $m(s) > (b-a) - \epsilon$ such that the sequence $\{f_n(x, y)\}$ converges uniformly on the set $S \subset R$, formed by the intersection of R with the lines through s parallel to the y-axis. Theorem II. Replace the assumption (b) of theorem I by (b'): for every y of $[c, d]$ one can extract a partial sequence from the given sequence $\{f_n(x, y)\}$, which converges almost everywhere in $[a, b]$. Then one can extract a partial sequence from $\{f_n(x, y)\}$, for which the conclusion of theorem I holds. From theorem II there follows immediately a third theorem concerning a sequence of functions $f_n(x, y)$ equicontinuous with respect to the two variables separately. *A. Rosenthal.*

Conti, Roberto. **Due criteri di convergenza uniforme per le successioni di funzioni monotone di due variabili in un rettangolo e nel piano.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 202–207 (1949).

Let the functions $f_n(x, y)$, $n = 1, 2, \cdots$, be monotone with respect to every variable separately in the rectangle R: $(a \le x \le b, c \le x \le d)$, i.e., for each $x = x_0 \varepsilon [a, b]$, $f_n(x_0, y)$ is to be a monotone function of y, and for each $y = y_0 \varepsilon [c, d]$, $f_n(x, y_0)$ is to be a monotone function of x. The author proves the following theorem. If this sequence $\{f_n(x, y)\}$ converges to a continuous function $f(x, y)$ in R, then $f(x, y)$ is also monotone with respect to every variable and the convergence is uniform. By an example it is shown that the assumption of continuity of $f(x, y)$ is essential for this theorem. The author proves also the uniform convergence of $\{f_n(x, y)\}$ under suitable assumptions if R is replaced by the total plane; hereby also certain conditions on the behavior of the functions at infinity have to be satisfied. *A. Rosenthal* (Lafayette, Ind.).

Zahorski, Z. **Sur la classe de Baire des dérivées approximatives d'une fonction quelconque.** Ann. Soc. Polon. Math. 21 (1948), 306–323 (1949).

The following theorem is proved for an arbitrary finite real function $f(x)$ (defined for every finite x). If the approximate derivative $f_{ap}'(x)$ exists at every point, except for a countable set, then $f_{ap}'(x)$ is a Baire function of class at most 3, relative to the set where it exists. The author states in a footnote that subsequently he has been able to strengthen his result, replacing 3 by 2. *A. Rosenthal.*

Ananthachar, V. S. **A sufficient condition for the existence of a derivative.** Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi Iaşi] 3, 775–776 (1948).

The author tries to prove that the right derivative of a continuous function $f(x)$ exists at a point x_0 if the equation $f(x) = mx$ has a finite number of roots in any small neighborhood $(x_0, x_0 + \delta)$ of x_0 for all real values of m. Unfortunately this assertion is wrong. Counterexample:

$$f(x) = 1 + \tfrac{1}{2}(x-1) \sin 1/(x-1)$$

at $x = y = 1$. *A. Rosenthal* (Lafayette, Ind.).

Császár, Ákos. **Sur une classe des fonctions non mesurables.** Fund. Math. 36, 72–76 (1949).

The author defines "fonction interne" and shows that every such function is either monotonic or non-Lebesgue measurable. This result is improved in a later paper [Acta Univ. Szeged. Sect. Sci. Math. 13, 48–50 (1949); these Rev. 10, 685]. *I. Halperin* (Kingston, Ont.).

✶Günther, N. **Sur les Intégrales de Stieltjes et Leurs Applications aux Problèmes de la Physique Mathématique.** Chelsea Publishing Company, New York, N. Y., 1949. iii+494 pp. $4.95.

The original was published in Trav. Inst. Phys.-Math. Stekloff 1, 1–494 (1932).

Lauwerier, H. A. **An elementary proof of the Arzela-Osgood-Lebesgue theorem.** Simon Stevin 26, 177–179 (1949). (Dutch)

Proof that $(R)\int f_n \to (R)\int f$ if $f_n \to f$ and $|f_n| \le \varphi$ on (a, b), where f_n, f, φ satisfy conditions somewhat more restrictive than necessary, using essentially Egoroff's theorem [Titchmarsh, The Theory of Functions, Oxford, 1932, p. 339]. *J. Korevaar* (Lafayette, Ind.).

Dubrovskiĭ, V. M. On equi-summable functions and on the properties of uniform additivity and equi-continuity of a family of completely additive set functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 341–356 (1949). (Russian)

This paper continues investigations of the author which have been described in earlier communications [Rec. Math. [Mat. Sbornik] N.S. 7(49), 167–178 (1940); 9(51), 403–420 (1941); 20(62), 317–329 (1947); Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 9, 311–320 (1945); Doklady Akad. Nauk SSSR (N.S.) 58, 737–740 (1947); 63, 483–486 (1948); these Rev. 2, 99; 3, 150; 9, 19; 7, 280; 9, 275; 10, 361]. For notation and terminology not explained here, see the reviews cited. New definitions are introduced, as follows. A Borel family $\mathfrak{M}$ is essentially divisible with respect to a basis M for $\mathfrak{M}$ if every $E \varepsilon \mathfrak{M}$ such that $M(E)>0$ can be divided into two disjoint subsets E_1 and E_2, both members of $\mathfrak{M}$, such that $M(E_1) \neq 0 \neq M(E_2)$. A set E of $\mathfrak{M}$ is said to be indivisible with respect to M if $M(E_1)$ or $M(E_2)$ is 0 for every division of E into disjoint sets E_1 and E_2, elements of $\mathfrak{M}$. A set $\{f_\lambda\}$, $\lambda \varepsilon \Lambda$, of real-valued functions on A is said to be equi-summable with respect to a measure M on $\mathfrak{M}$ if

$$\lim_{N \to \infty} \int_{E(N,\lambda)} (|f_\lambda(x)| - N) dM(x) = 0$$

uniformly with respect to the parameter λ, where $E(N, \lambda)$ is the set of x such that $x \varepsilon A$, $|f_\lambda(x)| > N$.

A number of theorems are proved, of which the following are typical. (1) If a family $\{\Phi_\lambda\}$ of completely additive set-functions on $\mathfrak{M}$ admits a basis, then it admits a regular basis. (2) If $\mathfrak{M}$ is essentially divisible with respect to a basis M for $\{\Phi_\lambda\}$, then it is essentially divisible with respect to any regular basis for $\{\Phi_\lambda\}$. (3) If M is any basis of the family of set-functions $\{\Phi_\lambda\}$ defined on $\mathfrak{M}$ such that $\mathfrak{M}$ is not essentially divisible with respect to M, then there exists a dissection of A (unique up to null-sets), $A = B \cup B'$, such that B is a countable union of pairwise disjoint sets each indivisible with respect to M, and such that every set of $\mathfrak{M}$ contained in B' is essentially divisible with respect to M. [Reviewer's note: this can be proved without assuming that M is a basis.] Also, this division is unique, up to null-sets, for all bases M. (4) If a family of set-functions $\{\Phi_\lambda\}$ on $\mathfrak{M}$ are uniformly additive, and admit a basis with respect to which $\mathfrak{M}$ is essentially divisible, then the functions Φ_λ are uniformly bounded on $\mathfrak{M}$. (5) If $\{\Phi_\lambda\}$ are a family of uniformly bounded and uniformly additive set-functions on $\mathfrak{M}$, and if M is any basis for $\{\Phi_\lambda\}$, then there exists a family of point-functions $\{f_\lambda(x)\}$ such that $\Phi_\lambda(E) = \int_E f_\lambda(x) dM(x)$ for all $E \varepsilon \mathfrak{M}$, and such that the functions $f_\lambda(x)$ are equi-summable with respect to M, $\mathfrak{M}$, and λ. A converse to (5) is also proved. (6) Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of finite-valued completely additive set-functions on $\mathfrak{M}$. If $\lim_{n \to \infty} \Phi_n(E)$ exists for every $E \varepsilon \mathfrak{M}$, then the set-functions are uniformly bounded on $\mathfrak{M}$. The paper closes with some remarks on abstract integral equations. *E. Hewitt* (Seattle, Wash.).

Kronrod, A. S. On a line integral. Doklady Akad. Nauk SSSR (N.S.) 66, 1041–1044 (1949). (Russian)

In this note the author's aim is to define for real-valued functions on J, the n-dimensional cube or sphere, methods of differentiation and integration which will recapture the validity of the common relations between these notions for functions of one variable. The derivation attaches to each function F with a total differential on J and to each point ξ of J a function f on J, $f(\eta) = F_\xi'(\eta)$, so that $f(\eta) = 0$, or

$f(\eta) = \pm |(\text{grad } F)(\eta)|$. The choice depends on the relationship between ξ and the component K of the level set E_t of F in which η lies (so $F(\eta) = t$) as follows: $f(\eta) = + |\text{grad } F|$ if (a) K separates its complement into two domains J_i such that $\xi \varepsilon J_1$, and (b) there exist continua R_i such that (i) $J_i \cup K \supseteq R_i \supset K$ and (ii) $F(\eta) \leqq t$ on R_1 and $F(\eta) \geqq t$ on R_2. If the inequalities in (ii) are reversed, $f(\eta) = - |\text{grad } F|$; $f(\eta) = 0$ otherwise.

For nonnegative functions f the "line integral" is defined as follows. Let Ω be the set where f is 0 and for each $\epsilon > 0$ let Ω_ϵ be the ϵ-neighborhood of Ω. For fixed ξ and ζ in J let L be a rectifiable curve connecting ξ to ζ and let

$$\int_\xi^\zeta f(\eta) d\eta = \lim_{\epsilon \to 0} \inf \int_L f(\eta) d\sigma,$$

where σ is arc length on L and the integral is taken over the part of L not in Ω_ϵ. When F is n-times differentiable on J it is stated that F is determined up to a constant by its derivative with arbitrary base-point ξ from the desired formula $\int_\xi^\zeta F'(\eta) d\eta = F(\zeta) - F(\xi)$. Also for continuous f and for $F(\zeta) = \int_\xi^\zeta f(\eta) d\eta$, it follows that $F_\xi'(\eta) = f(\eta)$ for almost all η in J. Certain elementary properties of the integral are mentioned including its nonadditivity. No proofs are given. *M. M. Day* (Urbana, Ill.).

Tolstov, G. P. Partial derivatives, curvilinear and multiple integrals. Uspehi Matem. Nauk (N.S.) 4, no. 3(31), 186–189 (1949). (Russian)

In this outline the author states a number of results proved in his doctoral dissertation. For example, if $f = u + iv$ is bounded in the plane region D and if the first partial derivatives exist everywhere and satisfy the Cauchy-Riemann equations almost everywhere in D, then f is analytic in D. Problems of equality of mixed partial derivatives of all orders up to m are studied. Another section considers functions whose iterated Denjoy integrals over all the rectangles in a given region are independent of the order of integration. *M. M. Day* (Urbana, Ill.).

Sunouchi, Gen-ichirô, and Utagawa, Masatomo. The generalized Perron integrals. Tôhoku Math. J. (2) 1, 95–99 (1949).

In the original definitions of the Perron integral O. Perron and O. Bauer employed only continuous major and minor functions. Saks [Theory of the Integral, 2d ed., Warszawa-Lwów, 1937, p. 201] gives a definition in which these conditions are dropped and shows that the scope of the integral remains unchanged [loc. cit., pp. 247–252]. The authors show that the corresponding approximate and Cesàro continuity properties of major and minor functions in the definitions of Burkill's AP-integral [Math. Z. 34, 270–278 (1931)] and CP-integrals [Proc. London Math. Soc. (2) 34, 314–322 (1932)] may be dropped to give integrals having the same properties as Burkill's integrals. The question as to whether or not the scope of Burkill's integrals is thereby increased is left open. *R. L. Jeffery* (Kingston, Ont.).

Malliavin, Paul. Majorantes et minorantes des fonctions simplement totalisables. C. R. Acad. Sci. Paris 229, 286–287 (1949).

A function $F(x)$, defined and continuous on $[a, b]$, belongs to the class $(\delta_{d,\tau} > -\infty)$ if for each perfect set P there exists a portion π of P such that at all points of π the lower right derived number $\delta_{d,\tau} F(x)$, with respect to π, exceeds $-\infty$. The function $H(x)$ is a weak simple major to the function $f(x)$ which is finite at all points of $[a, b]$ if: (1) $H(a) = 0$;

(2) $H(x)$ belongs to the class $(\delta_{4,\tau} > -\infty)$; (3) $H'_{ap}(x) \geqq f(x)$ at almost all points of $[a, b]$. The function $M(x)$ is a strong simple major to the same function $f(x)$ if: (1) $M(a)=0$; (2) $M(x)$ is continuous on $[a, b]$; (3) there exists a portion τ of P such that, at all points of τ, $\delta_{4,\tau} M(x) \geqq f(x)$. A strong simple major is likewise a weak simple major. There are corresponding definitions for simple minors. A measurable function $f(x)$ that possesses weak simple major and minor functions $H(x)$ and $G(x)$ is simply totalizable. A simply totalizable function possesses ϵ-adjoined strong simple major functions. R. L. *Jeffery* (Kingston, Ont.).

Graeub, Werner. **Transformation von Doppelintegralen.** Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **84**, 107–112 (1947).

Let T be a plane transformation of class C^1 defined on a region B_1. The author proves in a simple manner that if T is biunique on a bounded measurable region B which is strictly interior to B_1, then the image of B is measurable and its measure is equal to the integral over B of the absolute value of the Jacobian of T. The proof does not involve the usual assumption that the Jacobian is everywhere different from zero. R. G. *Helsel*.

Theory of Functions of Complex Variables

✻Green, S. L. **The Theory and Use of the Complex Variable. An Introduction.** Sir Isaac Pitman & Sons, Ltd., London, 1944. viii+136 pp. $3.75.

A brief introduction to certain topics at an elementary level. N. *Levinson* (Cambridge, Mass.).

✻Kneser, Hellmuth, und Ullrich, Egon. **Funktionentheorie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 189–242. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This report covers the advances of the theory of analytic functions during the indicated period in so far as they have attracted the attention of German mathematicians. The emphasis is rather heavily on meromorphic functions, conformal mapping and Riemann surfaces. References to German literature seem to be very complete. L. V. *Ahlfors* (Cambridge, Mass.).

Kneser, Hellmuth. **Über den Beweis des Cauchyschen Integralsatzes bei streckbarer Randkurve.** Arch. Math. **1**, 318–321 (1949).

The author modifies a proof of the strong form of the Cauchy integral theorem given by the reviewer [Bull. Amer. Math. Soc. **50**, 831–833 (1944); these Rev. **6**, 122] so that the Jordan curve theorem is needed only for simple polygons. L. H. *Loomis* (Cambridge, Mass.).

Deuring, Max. **Eine Bemerkung zum Cauchyschen Integralsatz.** Arch. Math. **1**, 321–322 (1949).

The author remarks on the use of the notion of rotation number (of a closed curve in the plane about a point not on the curve) in deriving the Cauchy integral theorem from the special case for the triangle. L. H. *Loomis*.

Herzog, Fritz, and Piranian, George. **Sets of convergence of Taylor series.** I. Duke Math. J. **16**, 529–534 (1949).

A subset M of the boundary C of the unit circle is a "set of convergence" if there is a Taylor series $\sum a_n z^n$ convergent for $|z| < 1$ and satisfying the conditions: (i) $\sum |a_n| = \infty$, (ii) $\lim a_n = 0$, (iii) $\sum a_n z^n$ converges for each $z\epsilon M$, (iv) $\sum a_n z^n$ diverges for each $z \epsilon C - M$. A set of convergence must be an $F_{\sigma\delta}$ set. In the opposite direction, Mazurkiewicz [Fund. Math. **3**, 52–58 (1922)] showed that any closed or open subset of C is a set of convergence. In the present paper, the authors show that any F_σ set is a set of convergence, and state that it will be shown in a later paper that the complement of any countable set is a set of convergence. If (iii) is strengthened to: (iii)' $\sum a_n z^n$ converges uniformly on M, then M is said to be a set of uniform convergence. It is shown that a set M is a set of uniform convergence if and only if it is closed. R. C. *Buck* (Madison, Wis.).

Schaeffer, A. C., Schiffer, M., and Spencer, D. C. **The coefficient regions of schlicht functions.** Duke Math. J. **16**, 493–527 (1949).

Let $\mathfrak{B}_n$ be the closed $2n$-dimensional region of points $(a_2, a_3, \cdots, a_{n+1})$ associated with the elements

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

which, when regular and schlicht in $|z| < 1$, form a family $\mathfrak{F}$. The coefficient problem for schlicht functions is that of determining the boundary $\mathfrak{S}_n$ of $\mathfrak{B}_n$. Using variational methods the authors are led to the theory of the characteristic curves of the partial differential equation of $\mathfrak{S}_n$ and Löwner's differential equation for schlicht functions associated with them. By integrating a system of ordinary differential equations curves on $\mathfrak{S}_n$ terminating in $(2, 3, \cdots, n+1)$ on $\mathfrak{S}_n$ are obtained. There is a second method where the curves obtained lie entirely in $\mathfrak{B}_n$, connecting the origin to the points of $\mathfrak{S}_n$. The dual relationship and mutual interdependence of the different methods of the theory of schlicht functions is brought out.

By a method of variations [see M. Schiffer, Amer. J. Math. **65**, 341–360 (1943); these Rev. **4**, 215] a neighboring element $f^\Delta(z)$ (near $f(z)$) of $\mathfrak{F}$ is obtained so that to each point $a \equiv (a_2, a_3, \cdots, a_{n+1})$ of $\mathfrak{B}_n$ a whole neighborhood of points $(a_2^\Delta, a_3^\Delta, \cdots, a_{n+1}^\Delta)$ is obtained by the formula

$$a_{k+1}^\Delta = a_{k+1} + \rho^2 \left\{ \sum_{\nu=1}^{N} C_\nu U_{k+1}(z_\nu) + \sum_{\nu=1}^{N} C_\nu^* V_{k+1}^*(z_\nu) \right\} + o(\rho^2),$$
$$k = 1, 2, \cdots, n.$$

The C_ν are arbitrary, $|z_\nu| < 1$ and a^* denotes the conjugate of a; $U_{k+1}(z)$ and $V_{k+1}(z)$ are known rational functions in $f(z)$, $f'(z)$, z and the coefficients $a_2, a_3, \cdots, a_{k+1}$. In general a full neighborhood ($2n$-sphere) of the point a will be obtained for sufficiently small ρ. However, if the rank of the matrix

$$M = \begin{Bmatrix} U_p(z_q) & V_p^*(z_q) \\ V_r(z_q) & U_r^*(z_q) \end{Bmatrix} \quad (p, r = 2, \cdots, n+1; q, s = 1, \cdots, n)$$

is $2n-1$ for all z_ν, the variation of the point a will lie in a $(2n-1)$-dimensional hyperplane. This restriction leads to a differential equation for $w = f(z)$:

$$(1) \qquad (zw^{-1}dw/dz)^2 P_n(w) = Q_n(z)$$

where P_n, Q_n are known rational functions involving the point a. Thus these functions $f(z)$ are analytic on $|z| = 1$ except for a finite number of singular points, and the image of the unit circle by $w = f(z)$ is bounded by a finite number of analytic arcs. Similar considerations when the rank of M is less than $2n-1$ leads to the conclusion that the associated schlicht functions $f(z)$ are algebraic functions.

Every point $a \subset \mathfrak{S}_n$ gives rise to a matrix M of rank less than $2n$. However, when a is not on $\mathfrak{S}_n$ it might show a singular behavior because of the special type of variation employed. Other general variational methods are used [see G. Julia, Ann. Sci. École Norm. Sup. (3) **39**, 1–28 (1922)] in conjunction with the above attack with the result that it is found that the boundary $\mathfrak{S}_n$ consists of points a belonging to functions which are schlicht in $|z| < 1$, satisfy (1) and map the unit circle onto the w-plane slit along analytic arcs whose differential equation is determined [see M. Schiffer, Proc. London Math. Soc. (2) **44**, 432–449 (1938)].

A partial differential equation for $\mathfrak{S}_n$ is determined from the observation that if a tip of a slit corresponds to $z = 1/k$ on $|z| = 1$ then $f'(1/k)$ vanishes. From (1) this implies that $Q_n(z)$ has a zero of at least order two at $1/k$. This fact gives two equations from which k may be eliminated, furnishing an equation between the coordinates of the point a on $\mathfrak{S}_n$ and the components of the normal vector at the point a with respect to $\mathfrak{S}_n$. The complicated nonlinear partial differential equation of the first order obtained is investigated by replacing it by a system of ordinary differential equations of the characteristics. These are found to be equivalent to the partial differential equation for $Q = Q_n(z, t)$:

$$(2) \qquad \frac{\partial Q}{\partial t} = z \frac{\partial Q}{\partial z} \frac{1+kz}{1-kz} + \frac{kz}{(1-kz)^2} Q,$$

where t is the parameter of the characteristic. A study of the variation of the zeros $z_j = z_j(t)$ of Q with t is made and in particular it is found that all the z_j on $|z| = 1$ remain on $|z| = 1$ and move toward k^{-1} as t increases, while the roots in $|z| < 1$ move further away from $|z| = 1$. The choice of the zero for the role of $1/k$ in general defines a characteristic direction at the point a and if Q has several zeros on $|z| = 1$ there are then several characteristics through a. A system of ordinary differential equations is set up for each of these characteristics. Only if we follow a characteristic in the "right" direction from a do we stay on $\mathfrak{S}_n$.

These considerations are generalized to the function

$$f(z)^\mu = \sum_{k=0}^\infty a_{k+\mu}^{(\mu)} z^{k+\mu}, \qquad a_\mu^{(\mu)} = 1,$$

where μ is any real number. Now $\mathfrak{B}_n^{(\mu)}$ denotes the region of points $(a_{1+\mu}^{(\mu)}, a_{2+\mu}^{(\mu)}, \cdots, a_{n+\mu}^{(\mu)})$ and $\mathfrak{S}_n^{(\mu)}$ is its boundary. The characteristic equations have a form independent of μ. It is shown how the recursion formulas connect $\mathfrak{B}_n^{(\mu)}$ with $\mathfrak{B}_n^{(1)} = \mathfrak{B}_n$ so that a knowledge of $\mathfrak{B}_n^{(\mu)}$ for large integers μ would lead to new information in the case $\mu = 1$. A characteristic curve on $\mathfrak{S}_n^{(\mu)}$ corresponds to one fixed slit-structure in the $w = f(z)$ plane. It is found that $|a_{n+\mu}^{(\mu)}|$ always has a local maximum at points corresponding to the Koebe function and, for all sufficiently large μ, has no other local maxima.   *M. S. Robertson* (New Brunswick, N. J.).

Ilieff, Ljubomir. Anwendung eines Satzes von G. M. Golusin über die schlichten Funktionen. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 1, 21–24 (1949).

Two theorems are proved. (I) If $f_k(z) = \sum_{n=0}^\infty a_{nk+1} z^{nk+1}$, $a_1 = 1$, is regular and univalent in $|z| < 1$, and if

$$|z_1|, \ |z_2| \le r < 1,$$

then

$$\frac{1-r^2}{(1+r^k)^{4/k}} \le \left| \frac{f_k(z_1) - f_k(z_2)}{z_1 - z_2} \right| \le \frac{1}{(1-r^k)^{4/k}(1-r^2)},$$

a generalization of the Verzerrungssatz. (II) The partial sums of the series for $f_2(z)$ are always univalent for $|z| < 3^{-\frac{1}{2}}$. This constant is best possible.   *A. W. Goodman.*

Iliev, Lyubomir. Application of a theorem of G. M. Goluzin on univalent functions. Doklady Akad. Nauk SSSR (N.S.) **69**, 491–494 (1949). (Russian)

Russian translation of the paper reviewed above.
          A. W. Goodman (Lexington, Ky.).

Rényi, Alfréd. On the coefficients of schlicht functions. Publ. Math. Debrecen **1**, 18–23 (1949).

The author shows that the Bieberbach conjecture $|a_n| \le n$, $n = 2, 3, \cdots$, for functions (1) $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$, regular and schlicht for $|z| < 1$, holds for a special class of schlicht functions. This class, first investigated by V. Paatero [see Ann. Acad. Sci. Fennicae. Ser. A. **37**, no. 9 (1933)], consists of those schlicht functions (1) which map the unit circle on a domain G of boundary rotation $\alpha \le 4\pi$, i.e., for $0 \le |z| < 1$

$$(2) \qquad \int_0^{2\pi} |\Re\{1 + zf''(z)/f'(z)\}| \, d(\arg z) \le \alpha \le 4\pi.$$

This end is accomplished by showing with the aid of a lemma of S. Banach [see Fund. Math. **7**, 225–236 (1925)] that the domain G is the limit of domains convex in one direction. For the class of functions convex in one direction the Bieberbach conjecture was earlier shown to be true [see M. S. Robertson, Amer. J. Math. **58**, 465–472 (1936)].

In case $\alpha \le 3\pi$ a sharper estimate for $|a_n|$ is obtained:

$$(3) \qquad |a_n| \le \prod_{k=2}^n (1 + (\alpha - 2\pi)/k\pi) \le n^{(\alpha-2\pi)/\pi}, \qquad n = 2, 3, \cdots,$$

which is sharp for $\alpha = 2\pi$ (convex functions) but not for $\alpha = 3\pi$. The reviewer believes that inequality (5) of the paper should read

$$\prod_{k=2}^n \left(1 + \frac{\alpha - 2\pi}{k\pi}\right) \le \exp\left\{\left(\frac{\alpha - 2\pi}{\pi}\right) \cdot \sum_{k=2}^n k^{-1}\right\} \le n^{(\alpha-2\pi)/\pi}.$$

          M. S. Robertson (New Brunswick, N. J.).

Goodman, Adolph-W. Sur les coefficients des fonctions p-valentes. C. R. Acad. Sci. Paris **228**, 1917–1918 (1949).

It is supposed that $f(z) = \sum_1^\infty b_n z^n$ is 2-valent in $|z| < 1$, and that the b_n are real. The following best possible estimates are stated:

$$(i) \qquad |b_3| \le \tfrac{4}{3}|b_1| + \tfrac{4}{3}|b_2|$$

if the map by $f(z)$ is convex;

$$(ii) \qquad |b_3| \le 5|b_1| + 4|b_2|,$$

if the map is starlike with respect to the origin. Similar results for p-valent functions are promised.
          W. W. Rogosinski (Stillwater, Okla.).

✱**Betz, Albert. Konforme Abbildung.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1948. viii+359 pp. 36 DM.

This is a book written by an engineer for engineers. In the author's opinion, conformal mapping is primarily of a geometrical nature and its connection with function theory is of a secondary character. In accordance with this belief, the first allusion to a relation between conformal mapping and complex variables is made on page 139. In the preceding part of the book, the author studies at great length some elementary conformal maps with the exclusive aid of geometrical and physical considerations; the arguments are largely heuristic. Quite apart from other objections, this

procedure may also be questioned on the grounds of economy since, in all these cases, lengthy arguments of dubious validity can be replaced (and made rigorous) by a few lines of the most elementary function theory.

The later parts of the book are written along more conventional lines. The mapping properties of the elementary functions and of elliptic functions are discussed in great detail; the Schwarz-Christoffel formula and Schwarz's procedure for mapping a half-plane onto a circular arc triangle are derived, although in a hydrodynamical disguise. Much care is devoted to the exposition of various techniques yielding conformal maps onto the outside of profiles of aerodynamical interest. This includes the discussion of a number of approximation methods, where again, in keeping with the spirit of the book, questions of convergence are passed over in silence.

In spite of all its mathematical shortcomings, the book is valuable for its detailed treatment of a large number of examples. Other valuable features are a discussion of applications of conformal mapping in the various branches of physics and last, but not least, a very considerable number of well-executed drawings. *Z. Nehari* (St. Louis, Mo.).

Komatu, Yûsaku. Conformal mapping of polygonal domains. Kōdai Math. Sem. Rep., no. 3, 7–10 (1949).

Continuing previous work [Proc. Imp. Acad. Tokyo **20**, 536–541 (1944); Jap. J. Math. **19**, 203–215 (1945); these Rev. **7**, 287], the author discusses extension of the Schwarz-Christoffel formula to conformal mappings of multiply-connected domains. Let Δ be a domain bounded by n circular polygons, and let $f(z)$ be a conformal mapping of a region D, bounded by n circles, upon Δ. It is stated that the Schwarzian derivative

$$\{f(z), z\} = f'''(z)/f'(z) - \tfrac{3}{2}(f''(z)/f'(z))^2$$

has certain automorphic properties with respect to the linear transformations associated with inversion in the circles bounding D. Similar results are given when D is a different canonical domain, such as the plane slit along n parallel segments, or the unit circle covered n times.
P. R. Garabedian (Berkeley, Calif.).

*Vidav, Ivan. **Kleinovi teoremi v teoriji linearnih diferencialnih enačb. Kleinsche Theoreme in der Theorie der linearen Differentialgleichungen.** Thesis, Akademija Znanosti in Umetnosti v Ljubljani, Matematično-Prirodoslovni Razred, 1941. 64 pp. (~~Croatian~~ Slovenian. German summary)

Let $-\infty < a < b < c < d < \infty$. The two accessory parameters of a Fuchsian differential equation of second order (all constants real) with the singularities a, b, c, d, ∞ are made subject to the requirement that the ratio w of two independent solutions be single-valued in the complex z-plane furnished with the slits $a < z < b$ and $c < z < \infty$. It is shown by elementary continuity considerations that there exist values of the accessory parameters such that the slit plane is mapped by w onto a schlicht domain.
Z. Nehari (St. Louis, Mo.).

Vietoris, Leopold. Zur Geometrie der ebenen analytischen Kurven. Anz. Akad. Wiss. Wien. Math.-Nat. Kl. **83**, 17–20 (1946).

For a curve C, represented parametrically by the real analytic functions $x = x(t)$ and $y = y(t)$ of the real parameter t, the author considers [H. A. Schwarz, Gesammelte Mathematische Abhandlungen, v. 1, p. 12; v. 2, pp. 66, 106, 151]

the analytic continuation of the function $z(t) = x(t) + iy(t)$ of the complex parameter t. With Schwarz, he calls points z, z^* inverse points with respect to C provided $z = z(t)$, $z^* = z(\bar t)$, where t and $\bar t$ are conjugate complex numbers. The operation of inversion depends only on the curve C, for it is independent of the choice of the parameter t. The author now continues the study of the geometry of the Riemann surfaces determined by the above inversely conformal transformations, defining and briefly discussing such entities as branch points, foci, poles, and essential singularities.
E. F. Beckenbach (Los Angeles, Calif.).

Heins, Maurice. The conformal mapping of simply-connected Riemann surfaces. Ann. of Math. (2) **50**, 686–690 (1949).

Démonstration du théorème fondamental de l'uniformisation pour une surface de Riemann ouverte, définie à la manière de Weyl-Radó [H. Weyl, Die Idee der Riemannschen Fläche, Teubner, Leipzig-Berlin, 1913; T. Radó, Acta Litt. Sci. Univ. Szeged. Sect. Sci. Math. **2**, 101–121 (1925)]. L'auteur souligne que sa démonstration ne fait aucun appel aux résultats de la théorie topologique des surfaces et, en particulier, à la triangulabilité. L'essentiel de la méthode consiste à faire usage du groupe des transformations linéaires qui transforment en lui même le cercle unité et à s'appuyer sur la démonstration de Perron pour l'existence de fonctions de Green.
S. Stoïlow (Bucarest).

Shirai, Tameharu. A remark on Riemann surface defined by M. S. Stoïlow. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **23**, 369–372 (1942).

Remarque sur la définition indiquée dans le titre, suivie d'un raisonnement complémentaire destiné à justifier la dite définition.
S. Stoïlow (Bucarest).

Moppert, Karl-Felix. Über eine gewisse Klasse von elliptischen Riemannschen Flächen. Comment. Math. Helv. **23**, 174–176 (1949).

The author constructs all rational functions $m = R(p)$ which map exactly $p = 0, 1, \infty$ onto $m = 0$, all whose zeros and one-points have multiplicity v_0, v_1, respectively, and which have no other multiple points. He studies their Riemann surfaces. There are exactly four such functions. If $m(z)$ is regular in $|z| < 1$ and has all its zeros with multiplicity v_0 and all its one-points with multiplicity v_1, then $m(z) = R[p(z)]$, where $p(z) \neq 0, 1, \infty$ in $|z| < 1$ and conversely.
W. K. Hayman (Providence, R. I.).

Pfluger, Albert. La croissance des fonctions analytiques et uniformes sur une surface de Riemann ouverte. C. R. Acad. Sci. Paris **229**, 505–507 (1949).

Let $\mathfrak{F}$ be an open Riemann surface and let

$$F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots$$

be a sequence of regions exhausting $\mathfrak{F}$ which have sufficiently smooth boundaries. Let w be analytic and single-valued on $\mathfrak{F}$ and set

$$D_n = \iint_{F_n} |dw/dz|^2 dx dy.$$

By use of suitable harmonic functions with boundary values 0 or 1, moduli $k_n > 0$ are associated with the sets $F_n - F_{n-1}$, and it is shown, using Schwarz's inequality, that (1) $D_n \geq D_0 \exp\{4\pi \sum_1^n k_i\}$. If $\mathfrak{F}$ possesses a Green's function g, one can take lines $g = \rho$ as the boundaries of the regions F_n. Let $H(\rho)$ be the largest of the periods of the conjugate

harmonic function to g about the components of the set $g = \rho$. Then the author's method yields the inequality

$$(2) \qquad D(\rho) = \iint\limits_{s \geq \rho} |dw/dz|^2 dx dy \geq D(0) e^{4\pi a(\rho)},$$

$$\alpha(\rho) = \int_\rho^\infty \{H(\xi)\}^{-1} d\xi.$$

From this follows for $M(\rho) = \max_{\rho = \rho} |w|$ the inequality

$$(3) \qquad M(\rho)^2 \geq C \int_\rho^\infty e^{4\pi a(\rho)} d\rho, \qquad C > 0,$$

from which a necessary condition for the existence of a non-constant bounded analytic function on $\mathfrak{F}$ is deduced. Thus the author obtains estimates from below upon the growth of the quantities $D(\rho)$ and $M(\rho)$ by a procedure reminiscent of Grötzsch's method of strips [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 80, 367–376 (1928)] and the Ahlfors distortion theorem [L. Ahlfors, Acta Soc. Sci. Fennicae. Nova Ser. A. 1, no. 9 (1930)].

P. R. *Garabedian* (Berkeley, Calif.).

Pfluger, Albert. Des théorèmes du type de Phragmén-Lindelöf. C. R. Acad. Sci. Paris 229, 542–543 (1949).

Let D be a domain of the z-plane bounded by a finite number of sufficiently regular curves and having $z = \infty$ as a boundary point; let $\Phi(r)$ be the angular measure of the largest arc of $|z| = r$ in D; and let $E(r)$ be the characteristic function of the angular projection of the boundary Γ of D upon the positive real axis. If $w(z)$ is analytic in D, $|w(z)| \leq 1$ at finite points of Γ, and $M(r) = \max |w(z)|$ for $|z| = r$, $z \in D$, then

$$(1) \qquad (\log M(R))^2 \geq C \int_1^R \exp \alpha(\rho) d \log \rho,$$

$$\alpha(\rho) = 2\pi \int_1^\rho \{\Phi(r)\}^{-1} E(r) d \log r, \qquad C > 0,$$

as is to be proved by the methods of the paper reviewed above. Let $\mathfrak{E}(r)$ be the characteristic function of the subset of E such that $\Phi(r) \geq \pi/x$ $(x \geq \frac{1}{2})$. Then one has the following extension of the Phragmen-Lindelöf principle:

$$(2) \quad \liminf_{r \to \infty} e^{-\pi \lambda(r)} \log M(r) > 0, \quad \lambda(r) = \int_1^r \mathfrak{E}(\rho) d \log \rho.$$

P. R. *Garabedian* (Berkeley, Calif.).

Haruki, Hiroshi. On Ivory's theorem. Math. Japonicae 1, 151 (1949).

Let z_j, $j = 1, 2, 3, 4$, be the vertices, in order, of a variable rectangle in the complex z-plane; and for an entire function $f(z)$ let $w_j = f(z_j)$. The author seeks the subclass F of the functions $f(z)$ for which $|w_3 - w_1| = |w_4 - w_2|$. The problem was suggested by the result of Ivory that the two diagonals of any curvilinear rectangle formed by four members of a system of confocal conics are equal. It is shown that the present class F consists of the functions $f(z) = az^2 + bz + c$, $f(z) = a \sin \alpha z + b \cos \alpha z + c$, $f(z) = a \sinh \alpha z + b \cosh \alpha z + c$, where a, b, c, and α are constants with α real.

E. F. *Beckenbach* (Los Angeles, Calif.).

Utz, W. R. On the decomposition of meromorphic functions. Revista Ci., Lima 50, 167–170 (1948).

The author extends a theorem of the reviewer [Trans. Amer. Math. Soc. 50, 1–14 (1941); these Rev. 3, 78] by generalizing the domain of definition of the functions con-

sidered from the unit circle to a simply-connected region R with a locally-connected boundary. [It seems to the reviewer that this extension can be proved more directly from the well-known facts that all the boundary points of R are accessible and that the mapping function for R is therefore continuous on the closed unit circle.] L. H. *Loomis*.

Anastassiadis, Jean A. Sur les combinaisons exceptionnelles des fonctions entières. Prakt. Akad. Athēnōn 15, 162–164 (1940). (Greek. French summary)

L'auteur veut exprimer les combinaisons exceptionnelles, homogènes ou non, des fonctions entières par rapport aux coefficients des fonctions données. Grâce aux théorèmes démontrés, on peut calculer exactement les combinaisons exceptionnelles d'un système $\{f(z)\}$ de ν fonctions, si les fonctions du système sont d'ordre fini.

Author's summary.

Anastassiades, Jean A. Sur les familles normales de fonctions entières et méromorphes d'ordre fini. Prakt. Akad. Athēnōn 19 (1944), 323–327 (1949). (Greek. French summary)

Pour les familles de fonctions entières d'ordre fini l'auteur arrive au théorème suivant. Si l'on donne p_n $(\leq k)$ et les inégalités $|c_i^{(n)}| \leq M$ $(i = 0, 1, \cdots, p_n)$, $|c_1^{(n)}| \geq \mu > 0$, toutes les fonctions entières d'ordre p_n, $f_n(z) = \sum_{m=0}^\infty c_m^{(n)} z^m$, ayant une valeur exceptionnelle dans un cercle de rayon R, forment une famille normale. On peut trouver une limite supérieure des modules ainsi que le type des fonctions de la famille. On obtient des résultats analogues pour les familles de fonctions méromorphes d'ordre fini.

From the author's summary.

Collingwood, E. F. Exceptional values of meromorphic functions. Trans. Amer. Math. Soc. 66, 308–346 (1949).

Les principaux résultats de ce mémoire ont été résumés dans quatre notes qui ont été analysées en détails ici [C. R. Acad. Sci. Paris 227, 615–617, 709–711, 749–751, 813–815 (1948); ces Rev. 10, 244, 363]. L'auteur y étudie les valeurs exceptionnelles ou non exceptionnelles d'une fonction $f(z)$ méromorphe dans un cercle $|z| < R$ (R fini ou infini) au point de vue individuel, et cherche des conditions pour qu'une valeur déterminée a soit normale, c'est-à-dire pour que le défaut

$$\Delta(a) = \limsup_{r \to R} m(r, a)/T(r) = 1 - \liminf_{r \to R} N(r, a)/T(r)$$

soit nul ($N(r, a)$ est la moyenne de Jensen du nombre des zéros de $f(z) - a$, $T(r)$ est la fonction caractéristique de Nevanlinna et $m(r, a) = T(r) - N(r, a) + O(1)$). Il généralise un théorème récent de H. L. Selberg fournissant une borne de $m(r, a)$ et en tire de nombreuses conséquences. L'idée directrice dans ces généralisations est de remplacer les conditions qualitatives par des conditions métriques appropriées; pour le détail on se reportera aux analyses citées. Notons que ces travaux de Collingwood ont déjà provoqué une note de Parreau [C. R. Acad. Sci. Paris 227, 1323–1325 (1948); ces Rev. 10, 442] où le défaut d'Ahlfors est étudié d'une façon analogue. G. *Valiron* (Paris).

Džrbašyan, M. M. On metric criteria for the completeness of the system of polynomials, with respect to weighted approximation. Doklady Akad. Nauk SSSR (N.S.) 66, 1037–1040 (1949). (Russian)

Extending earlier results attributed to A. L. Šaginjan [C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 425–428

(1937), and a later paper] and the author, a number of results are stated (without proof), including the following. Let $h(z)$ be a positive function in the circle $C: |z| < 1$, bounded away from zero on every closed subset of C, and with $\int\int_{|z|<1} h(z) dy dx < \infty$; let $H_2(h)$ be the class of functions $f(z)$, holomorphic in C, for which $\int\int_{|z|<1} h(z) |f(z)|^2 dy dx < \infty$. Then the system of polynomials is said to be complete in the class $H_2(h)$ if

$$\inf_{\{Q\}} \int\int_{|z|<1} h(z) |f(z) - Q(z)|^2 dy dx = 0,$$

where (here and later) $\{Q\}$ is the set of all polynomials. A sufficient condition for completeness in $H_2(h)$ is that (1) $h(z) \leq A h(rz)$, with $A > 0$, $r_0 \leq r < 1$, $|z| < 1$.

Let $h(z) > 0$ be continuous in $|z| \leq 1$ and bounded away from zero in an arbitrary circle $|z| \leq r < 1$; let $A(h)$ be the class of functions $f(z)$, holomorphic in $C: |z| < 1$ and fulfilling the conditions: (i) $\varphi(z) = h(z) f(z)$ is continuous in $|z| \leq 1$; (ii) wherever $h(e^{i\theta}) = 0$ there also $\varphi(e^{i\theta}) = 0$. If $h(z)$ satisfies inequality (1) (above) in $|z| \leq 1$, then the system of polynomials is complete in the class $A(h)$ in the sense $\inf_{\{Q\}} h(z) |f(z) - Q(z)| = 0$ ($|z| \leq 1$).

Let $P(r)$ be a mass function defined on $(0, \infty)$, having the representation $P(r) = P(1) + \int_1^r t^{-1} \omega(t) dt$ for $r \geq 1$, where $\omega(t)$ is nondecreasing and $\omega(t) \to \infty$ as $t \to \infty$; let $G\{\alpha_i\}$ be a simply-connected infinite region of the plane obtained by deleting a finite number of sectors $\Delta(\alpha_i)$ of angle π/α_i ($\frac{1}{2} < \alpha_i < \infty$; $i = 1, \cdots, m$), the sectors having no interior or boundary point in common; let $H_2(e^{-P(r)})$ be the class of functions $f(z)$, holomorphic inside $G\{\alpha_i\}$ and for which

$$\int\int_{G\{\alpha_i\}} e^{-P(r)} |f(z)|^2 dx dy < \infty, \qquad r = |z|.$$

The system of polynomials is complete in this class, in the sense

$$\inf_{\{Q\}} \int\int_{G\{\alpha_i\}} e^{-P(|z|)} |f(z) - Q(z)|^2 dx dy = 0,$$

if and only if $\int^\infty r^{-1-\alpha} P(r) dr$ diverges, where $\alpha = \max \{\alpha_i\}$ ($i = 1, \cdots, m$). Divergence of this integral is also necessary and sufficient for completeness in the class $A(e^{-P(r)})$ in the sense $\inf_{\{Q\}} e^{-P(|z|)} |f(z) - Q(z)| = 0$, $z \epsilon \bar{G}\{\alpha_i\}$. Here $A(e^{-P(r)})$ is the class of functions $f(z)$, holomorphic inside $G\{\alpha_i\}$ and satisfying: (i) $e^{-P(|z|)} f(z)$ is continuous in the closure of $G\{\alpha_i\}$; (ii) $\lim_{|z|\to\infty} e^{-P(|z|)} f(z) = 0$. *I. M. Sheffer.*

Džrbašyan, M. M. On the completeness of certain systems of analytic functions in infinite regions. Doklady Akad. Nauk SSSR (N.S.) **67**, 15–18 (1949). (Russian)

Let $D = D(\alpha)$ be the sector $-\pi \leq \beta < \arg z < \beta + \alpha \leq \pi$ and $H(\mu, p)$ ($\mu, p > 0$) the space of functions regular in D and of finite norm:

$$\|f\| = \left(\int\int_D \exp\{-\mu |z|^p\} |f(z)|^2 dx dy \right)^{\frac{1}{2}}.$$

Let $A(\mu, p)$ be the space of functions regular in D, continuous in $\bar{D}$, with finite norm

$$\|f\| = \sup_{z \in \bar{D}} \exp\{-\mu |z|^p\} |f(z)|$$

and such that $\exp\{-\mu|z|^p\} f(z) \to 0$ as $z \to \infty$ in $\bar{D}$. Let $\{\lambda_n\}$ be a sequence of positive numbers, $\lambda_{n+1} - \lambda_n \geq c > 0$. The fol-

lowing results are announced. (1) $\{z^{\lambda_n}\}$ ($|\arg z| \leq \pi$) is closed in $H(\mu, p)$ and in $A(\mu, p)$ if there is a nondecreasing $h(r)$ such that $\int h^p(r) r^{-2} dr = \infty$, $h(r) = O(r^{-\alpha} \exp\{2\sum_{\lambda_n < r} \lambda_n^{-1}\})$. The condition of the theorem is "in general" also necessary. In particular, $\{z^{\lambda_n}\}$ cannot be closed for integers λ_n unless $p \geq (2 - \alpha)^{-1}$. (2) Let $g(z)$ be an integral function of order ρ, type σ with $g^{(\lambda_n)}(0) = 0$, where the $\{\lambda_n\}$ satisfy the condition of theorem 1. Let $p > \rho$, $c_n = n |a_n|^{-\rho p/(p-\rho)}$, $d = (2\sigma\rho/\mu p)^{p/(p-\rho)}$, $B = \max_{0 \leq z \leq \frac{1}{2}\pi} x \cos x = b \cos b$. Then $\{g(a_n z)\}$ is closed in $H(\mu, p)$ and in $A(\mu, p)$, if one of the following conditions holds: (a) $\lim \inf c_n > d$, (b) $\lim \sup c_n > ed$, (c) the a_n lie in an angle of opening 2ω and $\lim \inf c_n > \omega d / \pi B$ when $\omega p \rho > b(p - \rho)$, $\lim \inf c_n > \{(p-\rho)d / \pi \rho p\} \sec \{\omega p \rho / (p - \rho)\}$ when $\omega p \rho \leq b(p - \rho)$. The conclusion also holds, if $\rho = p$, $|a_n| < (\mu/2\sigma)^{1/p} = R$ and the a_n have a limit point a with $|a| < R$. Results on the closure of $\{z^{\lambda_n}\}$ and $\{g(a_n z)\}$ are also given for the case that D is replaced by a curve starting and ending at infinity and satisfying suitable conditions. [See also the author's earlier paper, same Doklady (N.S.) **62**, 581–584 (1948); these Rev. **10**, 364.]

W. H. J. Fuchs (Liverpool).

Eweida, M. T. On the representation of integral functions by generalized Taylor's series. Proc. Math. Phys. Soc. Egypt **3**, 39–46 (1948).

The author considers the basic series associated with the operators $f^{(n)}(a_n)/n!$, $n = 0, 1, 2, \cdots$, where $|a_n| \leq kn^\alpha$ ($k > 0, \alpha > 0$). He shows, by using J. M. Whittaker's methods, that the basic series represents all entire functions of increase less than order $1/(1+\alpha)$, type $(1+\alpha)\{k^{-1} \log 2\}^{1/(1+\alpha)}$. He also shows by an example that the theorem to the same general effect given by Takenaka [Proc. Phys.-Math. Soc. Japan (3) **14**, 529–542 (1932)] gives an incorrect value for the type. *R. P. Boas, Jr.* (Providence, R. I.).

Makar, Ragy H., and Mursi, M. On the representation of functions by basic series of polynomials. Proc. Math. Phys. Soc. Egypt **3**, 47–55 (1948).

In the first part of the paper the authors consider a basic set $\{p_n(z)\}$ of polynomials and its "differential set" $\{p_n'(z)\}$ and "integral set" $\{1, \int_0^z p_n(t) dt\}$. Let D_n be the degree of the polynomial of highest degree in the expansion of z^n in terms of the $p_n(z)$, and suppose that (*) $\lim \sup D_n^{1/n} = 1$. Then the integral set is effective if and only if the original set is; if $p_0(z)$ is constant, the differential set is effective where the original set is. These results fail if (*) is not satisfied.

In the second part the authors suppose that $\{p_n(z)\}$ is a simple set, with zeros in $|z| < kn^\lambda$ ($\lambda \geq 0$) and $\lim \sup (\log r_n)/(\log n) = 0$, where r_n denotes the number of zeros of $p_n(z)$ other than $z = 0$. Then the basic series represents all entire functions of order less than $1/\lambda$, but not necessarily those of order $1/\lambda$. *R. P. Boas, Jr.*

✱**Petersson, Hans.** Elliptische Modulfunktionen und automorphe Funktionen. Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 243–275. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Petersson, Hans. Über die systematische Bedeutung der Eisensteinschen Reihen. Abh. Math. Sem. Univ. Hamburg **16**, 104–126 (1949).

In previous papers the author has investigated Eisenstein series of dimension $-r$ ($r \geq 2$):

$$G_{-r}(\tau, a_1, a_2, N) = \sum{}'(m_1\tau + m_2)^{-r}, \qquad (a_1, a_2) = 1,$$

where $m_i = a_i(N)$; for $r = 2$, convergence fails and the series must be modified by the introduction of a convergence factor due originally to Hecke. Most of the present paper is devoted to the restatement of the author's previous results; toward the end of the paper he extends his theorems to $r = 1$, and proves in particular [notation the same as in previous papers; cf., e.g., Math. Nachr. 1, 218–257 (1948); these Rev. 10, 525]: the Eisenstein series $G_{-1}(\tau, a_1, a_2, N)$ of level N and dimension -1 constitute a basis for $\mathfrak{R}_1$, the family of modular forms which are orthogonal to every entire cusp-form of the same level and dimension.

            J. Lehner (Philadelphia, Pa.).

Petersson, Hans. Ein Konvergenzbeweis für Poincarésche Reihen. Abh. Math. Sem. Univ. Hamburg 16, 127–130 (1949).

The author furnishes a new proof of Poincaré's theorem, viz., the series $\sum |cw+d|^{-r}$, $r > 2$, converges absolutely for $|w| < 1$, where c, d run over the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of a substitution group Γ which has the unit circle as limit circle. The new proof depends on some elementary facts of the hyperbolic geometry of the unit circle.

            J. Lehner (Philadelphia, Pa.).

Matsumoto, Toshizô. Iteration of elliptic functions. Mem. Coll. Sci. Univ. Kyoto Ser. A. 25, 27–29 (1947).

Let $F(z)$ be an elliptic function with the fundamental periods ω and ω'. The author shows that, in each period parallelogram, there exist a finite number of points ζ such that $F(\zeta) = \zeta + W$, where $W = m\omega + m'\omega'$, and m and m' are suitable integers. If $F_n(z)$ denotes the nth iterate of $F(z)$, it is shown that, in a certain neighborhood of such a point ζ, $\lim_{n \to \infty} F_n(z) = \zeta + W$.   *Z. Nehari* (St. Louis, Mo.).

Pic, G. Sur les groupes de substitutions linéaires qui laissent n points inchangés. Acta Bolyaiana 1, 81–95 (1947). (French. Romanian and Hungarian summaries)

The author studies groups of linear substitutions of the type

$$z_i' = \frac{a_{i1}z_1 + a_{i2}z_2 + \cdots + a_{in}}{a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nn}}, \quad i = 1, 2, \cdots, n-1; \|a_{ik}\| = 1,$$

which leave n given points invariant. It is shown that such groups afford numerous examples of p-groups (groups of prime power order).   *Z. Nehari* (St. Louis, Mo.).

Yûjôbô, Zuiman. A theorem on Fuchsian groups. Math. Japonicae 1, 168–169 (1949).

Let G be a Fuchsian group of linear transformations which leave the unit circle invariant, and denote by D its fundamental domain which contains $z = 0$. Let $[z]$ denote the point of D equivalent to z ($|z| < 1$), and let $E(\theta)$ be the set of points $[re^{i\theta}]$, where $0 \leq r < 1$. Finally, let z_n denote the points equivalent to $z = 0$. Extending an older theorem of Myrberg, the author proves that there are the following two alternatives: if $\sum_n (1 - |z_n|) = \infty$, then $E(\theta)$ is everywhere dense in D for almost all $e^{i\theta}$ on $|z| = 1$; if $\sum_n (1 - |z_n|) < \infty$, then $\lim_{r \to 1} |[re^{i\theta}]| = 1$ for almost all $e^{i\theta}$ on $|z| = 1$. This theorem was first announced by Tsuji [Jap. J. Math. 19, 155–188 (1944); these Rev. 7, 516]; however, Tsuji's proof was invalid, since it was based on a conjecture which had been shown by Myrberg [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 10 (1941); these Rev. 7, 516] to be false.   *Z. Nehari* (St. Louis, Mo.).

Combes, Jean. Fonctions uniformes sur une surface de Riemann algébroïde. C. R. Acad. Sci. Paris 229, 14–16 (1949).

In this note the author announces results concerning the existence of functions rational on an algebroid (nonalgebraic) Riemann surface F, which have prescribed zeros and poles. It is indicated briefly how the theory of automorphic functions and Myrberg's method of exhausting F by parts F_n of suitable algebraic surfaces lead to an existence theorem in which no restrictions (except the obvious one that the sets of zeros and poles have no limit points in F) are placed upon the zeros and poles. Thus the arithmetic properties of algebroid surfaces differ radically from those of (compact) algebraic surfaces: all ideals in the ring of holomorphic functions on F are principal, or the necessary and sufficient conditions of Abel and Jacobi in the theory of algebraic surfaces are not required. Furthermore it is mentioned that the field of functions rational on F is quasi-algebraically closed if the projection of F on the base plane is simply connected. Finally the author asserts that his results extend to certain surfaces F where the number of sheets of the approximating algebraic portions F_n increases with n.

            O. F. G. Schilling (Chicago, Ill.).

Kodaira, Kunihiko. On the existence of analytic functions on closed analytic surfaces. Kôdai Math. Sem. Rep., no. 2, 21–26 (1949).

This paper is a preliminary report on an attempt to generalise the classical existence theorem of analytic functions on Riemann surfaces to the case of the theory of functions of two complex variables. Only the main results and an outline of the proofs are given. The possibility of introducing a positive definite metric without torsion on a closed analytic (complex) surface is assumed, and the methods of harmonic integral theory are used. The reasoning of another paper [Ann. of Math. (2) 50, 587–665 (1949); these Rev. 11, 108] is used, and the main result established is the following. Let $\Gamma_1, \cdots, \Gamma_r$ be irreducible analytic curves on the surface, and let $f_{ip}(z_1, z_2)$ be a function of the local parameters z_1, z_2 analytic in the neighbourhood of a point p such that if $F(z_1, z_2)$ is any function analytic in the neighbourhood of p which vanishes on Γ_i then $F(z_1, z_2)/f_{ip}(z_1, z_2)$ is analytic in the neighbourhood of p. Further, let $m_1, \cdots, m_r$ be integers such that $\sum m_i \Gamma_i \sim 0$. Then there exists a function $f(z_1, z_2)$, unique save for a constant multiplier, such that in the neighbourhood of the point p

$$f(z_1, z_2) = U(z_1, z_2) \prod_i f_{ip}^{m_i}(z_1, z_2),$$

where $U(z_1, z_2)$ is analytic in the neighbourhood of p and does not vanish at p.   *W. V. D. Hodge.*

Hermes, Hans. Analytische Mannigfaltigkeiten in Riemannschen Bereichen. Math. Ann. 120, 539–562 (1949).

The author defines a complex analytic variety in complex projective space as a point set whose intersection with the neighborhood of every point of the surrounding space is an algebroid, that is, the locus of a finitely-based ideal of convergent power series, and he proves that it always admits a decomposition, virtually unique, into countably many irreducible ones. He defines an (unimbedded) variety as a topological space which can be covered by a system of local Euclidean algebroids and which obeys a certain regularity requirement about the interlocking of such neighborhoods. He shows that if it is separable then it is again decomposable into irreducible ones, and he has a theorem to the

effect that if a point set in such a variety is everywhere in space locally algebroidal, then it is a variety itself. Methodically the author operates a great deal with partial ordering to clarify the relations between "local" and "global" but this is only subsidiary to basic algebraic and analytic facts. A theory of complex analytic functions on general varieties is not given in the paper.   *S. Bochner* (Princeton, N. J.).

Degtereva, M. P. On some properties of sedenions. Doklady Akad. Nauk SSSR (N.S.) **67**, 965–967 (1949). (Russian)

The author continues the discussion [same Doklady (N.S.) **60**, 1491–1493 (1948); **61**, 13–15 (1948); these Rev. **10**, 245] of analytic functions in an associative and distributive algebra of order n. For an algebra with basic multiplication table of the form $e_i e_j = \gamma_{ij} e_k \neq 0$, k depending on i and j, with $e_1 = 1$ and $e_i^2 = 1$ or ± 1, some elementary properties are briefly developed, and conditions for right and left hand derivatives, analogous to the Cauchy-Riemann equations, are given. Application is made in particular to the algebra of sedenions ($n = 16$).   *E. F. Beckenbach.*

Dramba, Constantin. Sur une extension de la notion de fonction analytique. C. R. Acad. Sci. Paris **228**, 1626–1628 (1949).

The extension consists in dealing with functions

$$F = X_1 + \epsilon X_2 + \epsilon^2 X_3 + \cdots + \epsilon^{n-1} X_n,$$

where $X_1, \cdots, X_n$ are real-valued functions of the real variables $x_1, \cdots, x_n$, and $\epsilon^n + 1 = 0$. The corresponding system of first order partial differential equations for the X_i, which reduces for $n = 2$ to the Cauchy-Riemann equations of usual analytic function theory, is given. The real components X_i of an analytic function F all satisfy a certain nth order partial differential equation. For $n = 3$, with x, y, z instead of x_1, x_2, x_3, this equation is

$$U_{xxx} - U_{yyy} + U_{zzz} + 3U_{xyz} = 0.$$

The formula $e^{\epsilon x} = S_1(x) + \epsilon S_2(x) + \cdots + \epsilon^{n-1} S_n(x)$, which reduces to $e^{ix} = \cos x + i \sin x$ for $n = 2$, is mentioned.

J. B. Diaz (Providence, R. I.).

Theory of Series

✱Knopp, Konrad. Unendliche Zahlenfolgen. Limitierungsverfahren. Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 125–153. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

This report gives, with complete references and without proofs, an account of work on series and related topics done in Germany during the years 1939–1946. The account is well organized, and systematically shows how new results are related to older ones.   *R. P. Agnew* (Ithaca, N. Y.).

Markovitch, D. Quelques remarques sur les progressions arithmétiques. Bull. Soc. Math. Phys. Serbie **1**, 17–21 (1949). (Serbian. Russian and French summaries)

Suppose that in a sequence $\{a_n\}$ we have $\Delta^\nu a_n > 0$ ($\nu = 0, 1, \cdots, K$), $\Delta^{K+1} a_n = 0$ and put $S_n = a_1 + \cdots + a_n$. It is shown that $(n+m)/n < S_{n+m}/S_n < \binom{n+m}{K+1} \div \binom{n}{K+1}$.

W. Feller (Ithaca, N. Y.).

Hsu, L. C. Application of a symbolic operator to the evaluation of certain sums. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 139–149 (1948).

In an earlier paper [Ann. Math. Statistics **15**, 399–413 (1944); these Rev. **6**, 234] the author has treated $S_n = \sum f(x_1) \cdots f(x_n)$, $\sigma_n = \sum f_1(x_1) \cdots f_n(x_n)$, where the summations are over all nonnegative integral solutions of $x_1 + \cdots + x_n = m$, and $f, f_1, \cdots, f_n$ are polynomials. Here the results are extended to arbitrary functions. One formula is

$$S_n = (f(0) + \Delta f(0) \cdot \Delta + \cdots + \Delta^m f(0) \cdot \Delta^m)^n \binom{x}{m}_{x=m+n-1},$$

generalizing the Newton interpolation formula. The other, for σ_n, is much more complicated, but involves the same operator. Several corollaries and applications are given. [The author's use of the term "partition," where order is relevant, is misleading; "composition" seems preferable.]

N. J. Fine (Philadelphia, Pa.).

Wuyts, P. On splitting the terms in a convergent series. Simon Stevin **26**, 180–188 (1949).   (Dutch)

For each $n = 1, 2, \cdots$, let the term $a(n)$ of a convergent series $a(1) + a(2) + \cdots$ be split into a finite sum $a(n) = a(n, 1) + a(n, 2) + \cdots + (a_n, k_n)$. The "gesplitste reeks" $a(1, 1) + \cdots + a(1, k_1) + a(2, 1) + \cdots + a(2, k_2) + \cdots$ is convergent if and only if

$$\lim_{n \to \infty} \max_{1 \leq k \leq k_n} |a(n, 1) + \cdots + a(n, k)| = 0.$$

R. P. Agnew (Ithaca, N. Y.).

✱Vernotte, Pierre. Théorie et Pratique des Séries Divergentes. La Sommation des Séries Divergentes par l'Interpolation Idéale. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 207, xvii + 479 pp. (1947).

This book is an enormous collection of numbered definitions, observations, and calculations involving divergent series of real numbers. These numbered items have, according to the introduction, been accumulated by the author during the past thirty years. Unfortunately, the relations of these items to each other, to meaningfulness, and to truth are in very many cases exceedingly vague. The book takes no account of progress of the past fifty years in the theory of series. It is based on what the author considers to be fundamental ideas, but some of these ideas are so vague that they lead to contradictions and confusion. Thus the book cannot be recommended to anyone who is unprepared to formulate his own opinion of the validity of the contents of a book on series.

The significant work in the book has its roots in the formula

$$(1) \quad F(x) = u_0 - \frac{x}{1} u_1 + \frac{x(x-1)}{2!} u_2 - \frac{x(x-1)(x-2)}{3!} u_3 + \cdots,$$

and involves methods for evaluation of series which may be motivated and formulated as follows. Let $u_0 + u_1 + \cdots$ be a given series, convergent or divergent. The series in the right member of (1), whether it be convergent or divergent, makes one think of an interpolatory function $F(x)$ of the real variable x for which, when $f_0, f_1, \cdots$ are defined by the following equations, $F(0) = u_0 = f_0$, $F(1) = u_0 - u_1 = \Delta u_0 = f_1$, $F(2) = u_0 - 2u_1 + u_2 = \Delta_2 u_0 = f_2$, $\cdots$, $F(n) = \Delta_n u_0 = f_n$, $\cdots$. Moreover, when $x = -1$, the series in the right member of (1) becomes $u_0 + u_1 + u_2 + \cdots$. All this motivates the following definition of the value of the series $\sum u_n$. Let $F(x)$ be the interpolatory function, defined for $x > -1$, determined by an interpolation process P to satisfy the conditions

$F(n)=f_n$; then $\lim_{x\to-1+}F(x)$ is, if it exists, the value assigned to $\sum u_n$ by the Vessiot method V_P. For example, in the case of the series $1-1+1-1+\cdots$, we find that $f_n=2^n$ and any method of interpolation which yields $F(x)=2^x$ for the interpolatory function furnishes the classic value $\frac{1}{2}$ for the series.

The book contains many calculations involving the series

$$(2) \qquad 1!-2!+3!-4!+5!-6!+\cdots$$

and similar series. A typical procedure is to calculate 6 of the numbers f_n, take fourth roots of these numbers f_n, use graphical or numerical methods to obtain a function $\varphi(x)$ interpolating to these roots, and then use $[\varphi(-1)]^4$ for the value of the series. Such procedures yield, in the case of (2), various approximations to the value $.4036526\cdots$ obtained by Euler. Since a process of interpolation is required to determine a method V_P, the book gives extensive discussions of interpolation.

The "interpolation idéale" referred to in the title of the book is concerned with the problem of finding the "best" interpolatory function $f(x)$, defined for all real x greater than -1, such that $f(x)$ takes prescribed values for non-negative integer values of x. A method for obtaining interpolatory functions is "ideal" if, for each function $g(x)$ in a class which is usually presumed to contain linear functions $ax+b$, x^2, and other functions, it yields $g(f(x))$ for the interpolating function having the value $g(a_n)$ when $x=n$ whenever it yields $f(x)$ for the interpolating function having the value a_n when $x=n$. Thus the familiar calculation

$$x!=1\cdot2\cdot3\cdot\,\cdots\,\cdot x=\lim_{n\to\infty}\frac{1\cdot2\cdot3\,\cdots\,(x+n)}{(x+1)(x+2)\,\cdots\,(x+n)}$$
$$=\lim_{n\to\infty}\frac{n!\,n^x}{(x+1)(x+2)\,\cdots\,(x+n)}\frac{(n+1)(n+2)\,\cdots\,(n+x)}{n^x}$$
$$=\lim_{n\to\infty}\frac{n!\,n^x}{(x+1)(x+2)\,\cdots\,(x+n)},$$

which is obviously valid when x is a positive integer, produces in the last member an interpolation "la plus raisonnable" because a similar calculation starting with $(x!)^2$ leads to the square of the last member. On the other hand the method by which the continuous extension of the function

$$f(x)=\pi^{-1}\sin\pi x\left\{\frac{a_0}{x}-\frac{a_1}{x-1}+\frac{a_2}{x-2}-\frac{a_3}{x-3}+\cdots\right\}$$

is taken to be "the" interpolating function for which $f(x)=a_n$, is bad because, as computations for $x=\frac{1}{2}$ show, the interpolating function f_2 for which $f_2(n)=(1/n!)^2$ is not the square of the interpolating function f_1 for which $f_1(n)=1/n!$. While this idea of ideal interpolation seems useful, the reviewer can feel no confidence in the author's statements about existence of ideal methods of interpolation which provide functions $F(x)$ interpolating to prescribed values f_n, and about uniqueness of the function $F(x)$ when one or more such ideal methods exist. Certainly the author's alleged "ideal polynomial interpolation" can lead to bizarre results when he uses for the value of a series the number $[F(-1)]^n$ where $F(x)$ is the interpolatory polynomial of degree k for which $F(j)=f_j^{1/n}$, $0\leq j\leq k$. To illustrate this by a trivial example in which calculation is easy, let $k=2$, $f_0=1$, $f_1=0$ and $f_2=0$; then $F(-1)=3$ and the value, 3^n, of the series depends violently on the n which he selects for his calculation.   *R. P. Agnew* (Ithaca, N. Y.).

Vernotte, Pierre. **Sommation des séries divergentes par une simple considération de régularité.** C. R. Acad. Sci. Paris **228**, 1918–1920 (1949).

Dans son livre [voir l'analyse ci-dessus] l'auteur a introduit les notions de séries définissables et de l'interpolée idéale. En utilisant onze termes de la série $\sum_1^\infty(-)^{n-1}n!$ il arrive ainsi à la valeur extrêmement approchée 0.4036525 (contre $0.4036563\cdots$). Pour les séries de terme général $(-)^n n!$, resp. $n!$ il arrive aux valeurs approchées 0.33197 et -0.30.   *J. G. van der Corput* (Amsterdam).

Teghem, J. **Généralisation d'un théorème de M. Macphail sur la sommabilité d'Euler-Knopp.** Mathesis **58**, 53–57 (1949).

For each complex r, a series $u_0+u_1+\cdots$ with partial sums s_0, s_1, $\cdots$ is evaluable to σ by the Euler-Knopp method $E(r)$ if

$$\lim_{n\to\infty}\sum_{k=0}^n\binom{n}{k}r^k(1-r)^{n-k}s_k=\sigma.$$

Let $g(\theta)$ be a function of period 2π for which $0<g(\theta)\leq\infty$. Let C_g be the class of power series $\sum a_n z^n$ having analytic extensions along radial lines from the origin such that, for each angle θ, the open segment from the origin to $g(\theta)e^{i\theta}$ is free from singularities. When $\alpha=\arg(1-r^{-1})$ the condition $|1-r^{-1}|\cos(\alpha+\theta)+[|r|^{-2}-|1-r^{-1}|^2\sin^2(\alpha+\theta)]^{\frac{1}{2}}>1/g(\theta)$, $0\leq\theta\leq2\pi$, on r is necessary and sufficient to imply that $\sum a_n$ is evaluable E_r whenever $\sum a_n z^n$ belongs to the class C_g. This generalizes the known theorem to which it reduces when $g(\theta)$ is a positive constant.   *R. P. Agnew.*

Teghem, J. **Sur un mode de prolongement analytique par sommation.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 357–360 (1949).

Transform the series $\sum u_n$ to $\sum u_n'$ where

$$u_n'=(1-b)^{-1-n}(E-b)^n u_0$$

and $E^k u_0=u_k$. Then $\sum_{n\geq0}u_n$ is said to be summable $\{E, b\}$ if $\sum_{n\geq m}u_n'$ converges for some m, and to be summable $\{E, b\}^*$ if there is a convergent series $\sum v_n$ such that $\sum(u_n+v_n)$ is summable $\{E, b\}$. Let $C(\zeta, b)$ be the set of z such that $|z-b\zeta|<|1-b||\zeta|$. If $f(z)=\sum a_n z^n$ has an extension to a neighborhood of infinity, with at worst a pole there, then $\sum a_n z^n$ is summable $\{E, b\}$ to $f(z)$ for all z interior to $\bigcap C(\zeta, b)$, taken over all singular points ζ of f. If the moduli of the singularities of f do not have a finite limit point, then $\sum a_n z^n$ is summable $\{E, b\}^*$ to $f(z)$ for all z interior to $\bigcap[C(\zeta, b)\cup C(\zeta, 0)]$.   *R. C. Buck.*

Karamata, Jovan. **On an inversion of Cesàro's method of summing divergent series.** Glas Srpske Akad. Nauka **191**, 1–37 (1948). (Serbian)

If d_1, d_2, $\cdots$ is a suitable monotone increasing sequence of positive integers, then the Tauberian condition

$$(1)\qquad \lim_{\epsilon\to0+}\lim\inf_{n\to\infty}\ \min_{n\leq m\leq n+\epsilon d_n}(s_m-s_n)\geq0$$

is a reasonable modification of the "langsam abfallende" condition of R. Schmidt [Math. Z. **22**, 89–152 (1925)] to which it reduces when $d_n=n$. Tauberian theorems are proved which involve (1) and generalize known theorems involving the Schmidt condition. For example, if s_n is a real sequence for which

$$(2)\qquad \sigma_n=n^{-1}\sum_{k=1}^n s_k=s+o(n^{-1}d_n)$$

and (1) holds, then $\lim s_n=s$. There are similar theorems in which (2) is replaced by the function-to-function transfor-

mation $\sigma(x)=x^{-1}\int_0^x s(t)dt$ and (1) is replaced by the analogous condition on $s(x)$. There are more inclusive theorems involving the Cesàro sequence and integral transformations of positive integer orders, and the Riesz transformation $\sigma_k(x)=\sum_{n\leq x}(1-n/x)^k u_n$. *R. P. Agnew* (Ithaca, N. Y.).

Mandelbrojt, Szolem, et Agmon, Shmuel. Une généralisation du théorème Tauberien de Wiener. C. R. Acad. Sci. Paris **228**, 1394–1396 (1949).

Suppose that $K(x)$, $K_1(x)$ both belong to $L(-\infty, \infty)$, that

$$g(u)=\int_{-\infty}^{\infty} K(x)e^{-iux}dx, \quad g_1(u)=\int_{-\infty}^{\infty}K_1(x)e^{-iux}dx,$$

and that every real interval I contains a subset E_I which has a reducible frontier set [the authors report in a letter that this condition should replace the condition of Jordan measurability which is used incorrectly in their paper], is contained in the set of zeros of $g(u)$, and contains all the zeros of $g_1(u)$ in I. Under these conditions the following results are proved. (I) If $h(x)$ is bounded and

$$\lim_{y\to\infty}\int_{-\infty}^{\infty} K_1(y-x)h(x)dx=A\int_{-\infty}^{\infty}K_1(x)dx,$$

then

$$\lim_{y\to\infty}\int_{-\infty}^{\infty} K(y-x)h(x)dx=A\int_{-\infty}^{\infty}K(x)dx.$$

(II) If $\epsilon>0$, there exist rational numbers $\xi_1, \cdots, \xi_n$ and complex numbers $a_1, \cdots, a_n$ such that

$$\int_{-\infty}^{\infty} |K(x)-\sum_1^n a_r K_1(\xi_r+x)| dx <\epsilon.$$

(III) Every bounded solution $\phi(x)$ of the equation

$$\int_{-\infty}^{\infty} K_1(y-x)\phi(x)dx=0$$

is also a solution of

$$\int_{-\infty}^{\infty} K(y-x)\phi(x)dx=0.$$

The first and second results generalize Wiener's Tauberian theorem and are deduced from the third. *H. R. Pitt.*

Muracchini, Luigi. Su alcune proprietà di particolari serie doppie. Boll. Un. Mat. Ital. (3) **3**, 228–236 (1948).

The author studies the convergence of series of the form $\sum a_{m,n}/(n^s+m^s)$. Theorems with some analogy to those of convergence for series of the form $\sum a_n n^{-s}$ are given: for instance, the existence of "abscissae" of convergence and absolute convergence. *S. Mandelbrojt* (Paris).

Bellman, Richard, and Straus, Ernst G. Continued fractions, algebraic functions and the Padé table. Proc. Nat. Acad. Sci. U. S. A. **35**, 472–476 (1949).

The authors present, with sketches of proofs, a method for obtaining the rational approximants of Frobenius-Padé for power series expansions of algebraic functions, and hence the attendant continued fraction expansions for these functions. Examples given include the square root of a polynomial of degree 3, and the function $(1+x)(1-x^2)^{\frac{1}{2}}$. The practical applicability of the method is limited by the fact that the transcendental functions required, in general, for the uniformization of curves of genus greater than unity are not readily accessible. *H. S. Wall* (Austin, Tex.).

Fourier Series and Generalizations, Integral Transforms

✱**Denjoy, Arnaud.** Leçons sur le Calcul des Coefficients d'une Série Trigonométrique. Quatrième Partie. Les Totalisations. Solution du Problème de Fourier. Premier Fascicule: Les Totalisations. Deuxième Fascicule: Appendices et Tables Générales. Gauthier-Villars, Paris, 1949. Fasc. I, pp. 327–481, 1500 francs; Fasc. II, pp. 483–715, 2200 francs.

[For the preceding parts of the book, published in 1941, see these Rev. **8**, 260.] This is the concluding part of the book. As in the earlier volumes, the discussion of the trigonometric integral is interspersed with that of other cognate topics. Chapter VII is devoted to the theory of the general Denjoy integral (totalisation simple) and the reader familiar with the problem (through, say, Saks's Theory of the Integral [2d ed., Warszawa-Lwów, 1937]) will not fail to notice novelty in the presentation, mainly in the use of the totalization of series. Chapter VIII gives a very detailed discussion of the notion of integral treated as a functional, and much of the space is given to Stieltjes integrals with respect to abstract measures. Chapter IX is the most important one and deals with the trigonometric integral. The author constructs trigonometric series converging everywhere and illustrating the necessity of various conditions in the definition of the trigonometric integral, and in particular the fact that in the construction of this integral the transfinite induction may actually require all ordinals less than Ω. This chapter is of considerable interest to anyone interested in the theory of nonabsolutely convergent integrals. In appendices one finds a theory of the special Denjoy integral based on the notion of majorants and minorants. One also finds a criticism of what is usually called Perron's definition of integral. This is merely one aspect of the author's distrust of a certain type of mathematical reasoning. One may not share the author's views here in their entirety, and still be in agreement with him about the importance of constructive definitions in the theory of integrals.

A. Zygmund (Chicago, Ill.).

Mohanty, R. A criterion for the absolute convergence of a Fourier series. Proc. London Math. Soc. (2) **51**, 186–196 (1949).

It is familiar that the convergence of a general Fourier series, at a point t, can always be reduced to that of a cosine-Fourier series $\Phi(t)\sim\sum_1^\infty a_n \cos nt$ for $t=0$, where $a_0=0$. The following theorem on the absolute convergence of $\sum a_n$ corresponds to the Hardy-Littlewood test [Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 35]. Suppose that (i) $\Phi(t) \log (k/t)$ is of bounded variation in $(0, \pi)$, where $k>\pi$; (ii) $\sum |n^\delta a_n-(n-1)^\delta a_{n-1}| <\infty$, where $0<\delta<1$. Then $\sum|a_n|$ converges. The proof is curiously indirect. Condition (i) implies $|R, \exp(n^{1-\delta}), 1|$-summability (absolute Riesz-summability) of $\sum a_n$; and this together with (ii) implies the convergence of $\sum|a_n|$. *W. W. Rogosinski.*

Hsiang, Fu-Cheng. On Hardy-Littlewood's convergence criterion for Fourier series. Acad. Sinica Science Record **2**, 270–271 (1949).

The author offers a new proof of the Hardy-Littlewood test for the convergence of Fourier series [Ann. R. Scuola Norm. Super. Pisa (2) **3**, 43–62 (1934)]. The main lemma of the proof is, however, incorrect within the limits required for the result. *A. Zygmund* (Chicago, Ill.).

Hyslop, J. M. Note on a group of theorems in the theory of Fourier series. J. London Math. Soc. **24**, 91–100 (1949).

Let $\varphi(t)$ be an even, periodic, integrable function, and

$$\varphi_\alpha(t) = \alpha t^{-\alpha} \int_0^t (t-u)^{\alpha-1}\varphi(u)du, \quad \alpha>0; \quad \varphi_0(t)=\varphi(t).$$

Let $C_\beta(t)$ be the Rieszian mean of order β and type n of the Fourier series of φ. The theorems of this paper, which generalize some previously known results, are as follows. (I) If $\alpha>0, 0\leqq\rho<\alpha, 0\leqq\rho<1$, and if as $x\to\infty$, $C_\alpha(x)=o(x^{-\rho})$, then, as $t\to0$, $\varphi_\beta(t)=o(t^\rho)$ for $\beta>\alpha+1-\rho$ and $\varphi_\beta(t)=o(t^\rho \log t^{-1})$ for $\beta=\alpha+1-\rho$. (II) If $\alpha>0, 0\leqq\rho<\alpha, 0\leqq\rho<1$ and if, as $x\to\infty$, $C_\alpha(x)=o(x^{-\rho}\log x^{-1})$, then, as $t\to0$, $\varphi_\beta(t)=o(t^\rho)$ for $\beta=\alpha+1-\rho$.

K. *Chandrasekharan* (Bombay).

Prachar, Karl, und Schmetterer, Leopold. Über die Euler'sche Summierung Fourier'scher Reihen. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **85**, 33–39 (1948).

The authors establish that the Lebesgue constants L_n of the Euler summability method (E, q) are of exact order $\log n$ for $q=1, 2$ and hence deduce the existence of continuous functions whose Fourier series are not summable (E, q). The latter fact is also noted by Hardy [Divergent Series, Oxford, 1949, p. 364; these Rev. **11**, 25].

P. *Civin* (Eugene, Ore.).

Nikolaev, V. F. On operations assigning polynomials to functions. Doklady Akad. Nauk SSSR (N.S.) **68**, 9–11 (1949). (Russian)

Suppose that $U_n{}^m(f)$ is a linear operation mapping the Banach space $\bar{C}$ of continuous 2π-periodic functions into itself in such a way that $U_n{}^m(f)$ is a trigonometric polynomial of degree not exceeding $n+m$ and that all trigonometric polynomials of degree not exceeding n remain unaltered. By elementary computations it is shown that

$$(1) \qquad \|U_n{}^m\| \geqq \left| \sum_{k=1}^n b_k \right| /2\|T\|$$

for any $T(\theta)=\frac{1}{2}a_0+\sum_{k=1}^n a_k \cos (m+k)\theta + b_k \sin (m+k)\theta$. Some consequences of (1) are deduced. (i) If, for a set of $U_n{}^m$, $n\to\infty$ and $m=o(n)$, then the norms $\|U_n{}^m\|$ are unbounded. (ii) The partial sums of a trigonometric series with coefficients a_n, b_n cannot be uniformly bounded unless $\sum_1^n b_k=O(\log n)$. [The topic of the note is related to a theorem of Lozinskiĭ, same Doklady (N.S.) **61**, 193–196 (1948); these Rev. **10**, 188.]

G. G. *Lorentz* (Toronto, Ont.).

Oswald, Jacques. Sur quelques propriétés des signaux à spectre limité. C. R. Acad. Sci. Paris **229**, 21–22 (1949).

The author considers signals with bounded spectra [see also Weston, Philos. Mag. (7) **40**, 449–453 (1949); these Rev. **10**, 552], that is, functions $x(t)$ whose Fourier transforms vanish outside a finite interval. Such a function is determined by its values at integral points, and a filter, corresponding to the multiplication of its Fourier transform by a given function, thus is determined by a transformation matrix. This matrix is evaluated for translation and derivation operators.

J. L. *Doob* (Ithaca, N. Y.).

Jacobson, A. W. A generalized convolution for finite Fourier transformations. Bull. Amer. Math. Soc. **55**, 804–809 (1949).
Jacobson, A. W. Solution of steady state temperature problems with the aid of a generalized Fourier convolution. Quart. Appl. Math. **7**, 293–302 (1949).

Let

$$S\{S\{F\}\} = \int_0^\pi \int_0^\pi F(x, y) \sin mx \sin nydxdy$$

be the repeated finite sine transform of $F(x, y)$, let $F_1(x, y)$ be the odd periodic extension (period 2π) of $F(x, y)$ with respect to x and an odd extension with respect to y, and define

$$F^*(x) = -\int_{-\pi}^\pi F_1(x-y, y)dy$$

as the generalised convolution of F. The generalised convolution theorem is $S\{S\{F\}\}=\frac{1}{2}C\{F^*\}$, where $m=n$ and C is the finite cosine transformation. There are similar convolution theorems for SC, CS, CC, but the parity of F_1 with respect to x and y is different in each case. If $F(x, y)=G(x)H(y)$, we have the well-known convolution theorems of the finite Fourier transformation.

With the aid of the generalised convolution a fairly general boundary value problem can be resolved into a problem with simpler boundary conditions and source function. In this problem V is a function of $x, y_1, \cdots, y_k$, the region is a cylinder whose generators are parallel to the x-axis, Q is a point on the curved boundary of the cylinder, and the bases are $x=0$ and $x=\pi$: $L[V]$ is a second-order differential operator in the y_i, $\lambda[V]$ the combination which appears in the boundary condition on the curved boundary. The partial differential equation is $V_{xx}+L[V]=F(x, y_i)$, the boundary conditions may be briefly indicated as $\lambda[V(x, Q)]=G(x, Q)$, $V(0, y_i)=H(y_i)$, $V(\pi, y_i)=K(y_i)$; F, G, H, K are given functions. The author constructs an auxiliary boundary value problem for a function $U(x, x', y_i)$, consisting of the partial differential equation

$$U_{xx}+L[U]=(1-x/\pi)F(x', y_i)$$

and boundary conditions $\lambda[U(x, x', Q)]=(1-x/\pi)G(x', Q)$, $U(0, x', y_i)=(1-x'/\pi)H(y_i)$, $U(\pi, x', y_i)=K(y_i)$. He then shows that

$$V(x, y_i) = \frac{1}{2}(\partial/\partial x)\int_{-\pi}^\pi U_1(x-x', x', y_i)dx',$$

which is a generalisation of Duhamel's integral.

In the first paper the generalised convolution theorem is proved for integrable functions, and the application to the above general boundary value problem is explained. In the second paper the author investigates certain special functions which often occur in solving problems of the indicated type, and discusses in more detail some two- and three-dimensional problems relating to Laplace's equation.

A. *Erdélyi* (Pasadena, Calif.).

Zoller, K. Die Entzerrung bei linearen physikalischen Systemen. Ing. Arch. **15**, 1–18 (1944).

The excitation function $E(t)$ and the output function $A(t)$ of a linear physical system satisfy an equation $L_a[A]=L_e[E]$, where L_a and L_e are unspecified linear homogeneous functional transformations. Conditions of continuity and boundness are imposed upon the functions and their transforms. The transition function $f(t)$ of the system is the output when the excitation is the unit step function, zero for $t<0$, unity for $t>0$. Then $A(t)=(d/dt)\int_0^t f(t-\tau)E(\tau)d\tau$, under certain

additional hypotheses on the operators and the function $E(t)$. A theory of systems that are linear in this limited sense is presented with the aid of the theory of Laplace transformations and integral equations. An order of inertia of the system is introduced and used. The existence, uniqueness and determination of $E(t)$ from $A(t)$ and $f(t)$, or from $A(t)$ and certain other transition functions, are examined. A simple graphical scheme for computing values of convolution (Faltung) integrals is presented as an aid in using some of the formulas. Special types of linear operators that satisfy the additional hypotheses made here are pointed out.
R. V. Churchill (Ann Arbor, Mich.).

Schwartz, Laurent. Généralisation de la notion de fonction et de dérivation. Théorie des distributions. Ann. Télécommun. **3**, 135–140 (1948).

Exposition of the author's theory [cf. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 57–74 (1946); **23**, 7–24 (1948); these Rev. **8**, 264; **10**, 36].

Kawata, Tatsuo, and Udagawa, Masatomo. On infinite convolutions. Kōdai Math. Sem. Rep., no. 3, 15–22 (1949).

The authors prove known theorems or add slight refinements to known theorems concerning the convergence of sequences of distribution functions (in particular, of infinite convolutions) and the continuity of the limit distribution function (when it exists). Several of the theorems, given without references, appear in papers by E. R. van Kampen and A. Wintner [Amer. J. Math. **59**, 635–654 (1937)] and E. R. van Kampen [Amer. J. Math. **62**, 417–448 (1940); these Rev. **1**, 209]. *P. Hartman* (Baltimore, Md.).

Chao, Robert F. H. Power series transform. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 122–138 (1948).

By and large, this is an exposition of the well-known method of generating functions. In the expansion $\phi(z) = \sum_{k=0}^{\infty} F_k z^k$, $\phi(z)$ is regarded as the "power series transform" of F_k, and F_k as the inverse power series transform of $\phi(z)$. The author investigates the operational rules of this transformation, lists a number of transform pairs, and applies his results to the solution of linear difference equations with constant coefficients, to a discussion of Newton's interpolation formula, to an investigation of the Laplace transforms of jump functions, and to derive certain relations for Stirling numbers. *A. Erdélyi* (Pasadena, Calif.).

Batschelet, Eduard. Die Operatorenmethode von L. Fantappiè und die Laplace-Transformation. Comment. Math. Helv. **22**, 200–214 (1949).

Der Autor stellt sich die Aufgabe das Verfahren der Laplace-Transformation mit der Operatorenmethode von Fantappiè zu vergleichen; es werden nur die einseitig unendlichen Laplace-Transformationen und die Funktionen des Operators $I = \int_{t_0} f(t) dt$ betrachtet und die allgemeineren Operatoren von Fantappiè in der Einleitung kurz erörtert. An Hand der hyperbolischen Differentialgleichung

$$u_{xy} + a(x)u_x + b(x)u_y + c(x)u = f(x, y),$$

welche mit beiden Methoden gelöst wird, werden die beiden Verfahren diskutiert. Dabei wird besonders hervorgehoben, dass die Operatoren $G(I)$ sich immer durch eigentliche Integrale mit endlichem Integrationsweg darstellen lassen und daher für bekannte und unbekannte Funktionen nur die bei Differentialgleichungen üblichen Voraussetzungen verlangen. Abschliessend wird die enge "formale" Verwandtschaft der beiden wesentlich verschiedenen Verfahren gezeigt. *H. G. Haefeli* (Boston, Mass.).

Agmon, Shmuel. Sur un problème de translations. C. R. Acad. Sci. Paris **229**, 540–542 (1949).

Let $K_0(x)\epsilon L^p(0, 1)$, $1 \leq p < \infty$, and $|K_0(x)| > 0$ on a set of positive measure in every interval $(0, \delta)$, $\delta > 0$; put $K_0(x) = 0$ outside $(0, 1)$. Let $\{\xi_n\}$ be a dense sequence on $(0, 1)$. Then the sequence $\{K_0(x - \xi_n)\}$ spans L^p. The proof is by means of entire functions. *R. P. Boas, Jr.* (Providence, R. I.).

Motzkin, Th. Approximation by curves of a unisolvent family. Bull. Amer. Math. Soc. **55**, 789–793 (1949).

It is well known that the polynomial $P(x)$ of degree $n-1$ having the smallest deviation $d = \max |f(x) - P(x)|$ from a given function $f(x)$ continuous for $-1 \leq x \leq 1$ has the characteristic property that $f(x) - P(x)$ assumes the values $\pm d$ at $n+1$ points with alternating signs. The main result of the present paper is an extension of this theorem to the case where the approximating function is taken from a very general family of n-parameter functions. *L. Fejes Tóth.*

✳**Maak, Wilhelm.** Fastperiodische Funktionen. Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 155–158. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

The author defines a function $F(P)$ on the surface of a sphere to be almost periodic if, for a given $\epsilon > 0$, one can find finitely many disjoint and exhaustive subsets $\mathfrak{T}_i$ of the surface of the sphere so that for P, Q in $\mathfrak{T}_i$ and c an arbitrary element of the 3-dimensional rotation group G, $|F(cP) - F(cQ)| < \epsilon$. By mapping the left cosets of G_0 (rotations about some fixed axis) onto the surface of the sphere in a one-to-one fashion, the almost periodic functions on the sphere become almost periodic functions on the group G which are constant on the cosets of G_0, and these functions form a left-module, i.e., if $f(x)$ is such a function on G, then $f(cx)$ is one also. The fundamental decomposition theorem for modules leads to the conclusion: every almost periodic function $F(P)$ on the surface of the sphere can be uniformly approximated by sums $\sum F_i(P)$ when the $F_i(P)$ belong to irreducible modules. The author asserts that results in v. d. Waerden's "Die Gruppentheoretische Methode in der Quantenmechanik" [Springer, Berlin, 1932] imply that every module is spanned by spherical harmonics, and thus one concludes that every almost periodic function on the sphere is continuous. The converse is obvious.
B. Gelbaum (Minneapolis, Minn.).

Jessen, Børge. On the proofs of the fundamental theorem on almost periodic functions. Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 8, 12 pp. (1949).

The author gives a new simplified proof of the Parseval equation for Bohr almost periodic functions based on the Parseval equation for Fourier transforms. Two variations of de la Vallée Poussin's proof of the uniqueness theorem are also given. *R. H. Cameron* (Minneapolis, Minn.).

Polynomials, Polynomial Approximations

✳**Marden, Morris.** The Geometry of the Zeros of a Polynomial in a Complex Variable. Mathematical Surveys, No. 3. American Mathematical Society, New York, N. Y., 1949. ix+183 pp. $5.00.

L'objet de ce livre est de rassembler les divers renseignements qu'on peut obtenir sur les zéros d'un polynome dont

les coefficients dépendent d'un certain nombre de paramètres variables, lorsque certains de ces paramètres ont des valeurs données ou varient dans des régions fixées du plan complexe. L'auteur laisse de côté les principes "qualitatifs" généraux relatifs à ces questions, pour s'attacher à traiter de façon aussi précise que possible chacun des problèmes qu'il envisage, en cherchant chaque fois à donner les meilleurs résultats explicites qui s'y rapportent.

Les 10 chapitres se divisent en deux parties: dans la première (qui comprend les 6 premiers chapitres), les polynomes considérés s'expriment à l'aide d'autres polynomes supposés connus, et il s'agit de déterminer la position de leurs racines en fonction de celles des polynomes donnés; la seconde partie traite de la position des racines en fonction des coefficients du polynome que l'on considère.

Les trois premiers chapitres sont consacrés à l'étude des zéros de la dérivée $f'(z)$ d'un polynome $f(z)$, dont les zéros sont supposés connus. L'auteur donne diverses interprétations (notamment des interprétations mécaniques) des zéros de $f'(z)$; les principaux théorèmes de ces chapitres sont le théorème de Lucas, d'après lequel les zéros de $f'(z)$ sont contenus dans tout polygone convexe contenant les zéros de $f(z)$, et le théorème de Jensen, qui, dans le cas où f a ses coefficients réels, montre que les zéros imaginaires de f' sont dans les cercles ayant pour diamètres les segments joignant les couples de zéros conjugués de f.

Le chapitre IV débute par l'énoncé du théorème de Grace: si deux polynomes $f(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k$, $g(z) = \sum_{k=0}^n \binom{n}{k} b_k z^k$ sont apolaires, c'est-à-dire que l'on a la relation bilinéaire $\sum_{k=0}^n (-1)^k \binom{n}{k} a_k b_{n-k} = 0$, alors tout domaine circulaire contenant tous les zéros de l'un contient au moins un zéro de l'autre [le rapporteur regrette que l'invariance de la notion d'apolarité par homographie n'ait pas été mentionnée]. Ce résultat est appliqué dans ce chapitre et les deux suivants à de nombreux problèmes sur les zéros de combinaisons linéaires de polynomes et de leurs dérivées, le plus souvent sous la forme équivalente commode due à Walsh: si $\phi(z_1, \cdots, z_n)$ est symétrique, et linéaire par rapport à chacun des z_k, alors lorsque les a_k $(1 \leq k \leq n)$ sont dans un domaine circulaire, il existe un point ζ de ce domaine tel que $\phi(\zeta, \cdots, \zeta) = \phi(a_1, \cdots, a_n)$. En particulier on obtient ainsi au chapitre V le domaine de variation des zéros de la dérivée d'un quotient de deux produits de polynomes, chacun des facteurs de ces produits étant assujetti à avoir ses zéros dans une région circulaire donnée. Au chapitre VI sont obtenus des bornes pour $p-1$ zéros de la dérivée de $f(z)$, lorsqu'on sait que $f(z)$ a p zéros dans un cercle donné, les autres étant arbitraires.

Le chapitre VII énumère diverses limites supérieures pour le module de tous les zéros de $f(z)$, en fonction des modules des coefficients de f; diverses précisions peuvent être apportées à ces résultats quand on tient compte des arguments des coefficients, ou du fait que le polynome f est lacunaire. Le chapitre VIII est consacré au problème de Landau-Montel, c'est-à-dire à la détermination du cercle de centre 0 où se trouvent toujours p zéros de $a_0 + a_1 z + \cdots + a_n z^n$, lorsque $a_0, a_1, \cdots, a_{p-1}$ et un coefficient a_{p+h} sont fixés, les autres étant arbitraires: on y prouve en particulier que si le polynome n'a qu'un nombre fixe k de coefficients non nuls (son degré n pouvant être arbitrairement grand), le rayon du cercle en question est borné par un nombre ne dépendant que de k (et des p coefficients fixés).

Enfin, les deux derniers chapitres exposent les critères (dûs en substance à Hermite, Hurwitz et I. Schur) permettant par des calculs de déterminants, de trouver le nombre des racines d'un polynome donné, situées dans un domaine circulaire donné.   *J. Dieudonné* (Nancy).

Marden, Morris. On the zeros of rational functions having prescribed poles, with applications to the derivative of an entire function of finite genre. Trans. Amer. Math. Soc. 66, 407–418 (1949).

The principal result is the following theorem. Let $F(z) = \sum_{j=0}^{p-1} A_j z^j + \sum_{j=1}^n m_j (z - z_j)$, where the A_j are arbitrary complex numbers and the m_j are complex numbers, i.e., $F(z)$ is a rational function with a pole of order $p-1$ at infinity and simple poles at the n finite points z_j with residues m_j such that $a \leq \arg m_j \leq a + \mu \leq a + \pi$, $j = 1, 2, \cdots, n$. Let K be the smallest closed convex region containing all the z_j. Then $F(z)$ has at most p zeros (counted with their multiplicities) exterior to the closed region $S(K, \psi)$ comprised of all points from which K subtends an angle of at least $\psi = (\pi - \mu)/(p+1)$. A similar theorem is established for the zeros of meromorphic functions of the form $\phi(z) = P(z) + [m_0/(z - z_0)] + \sum_{j=1}^n m_j \phi(z, z_j)$, where $P(z)$ is an arbitrary polynomial of degree p and

$$\phi(z, c) = (z - c)^{-1} + \sum_{k=q}^{p-1} z^k / c^{k+1},$$

$q \leq p$. This theorem is applied to the derivative of entire functions of genre p,

$$E(z) = e^{P(z)} z^r \prod_{j=1}^\infty \{[1 - (z/z_j)] \exp \sum_{k=1}^{p_2} (1/k)(z/z_j)^k\},$$

where $P(z)$ is a polynomial of degree p_1 and $p = \max(p_1, p_2)$. The location of the zeros of $E'(z)$ relative to the zeros of $E(z)$ is determined independently of $P(z)$. In particular, the Lucas theorem on the zeros of the derivative of a polynomial is generalized to entire functions.   *E. Frank.*

Obrechkoff, N. Sur les zéros des dérivées des fonctions rationnelles. C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 1, 5–8 (1948).

The following theorems are proved concerning the zeros of the rational function $f(z) = P(z)/Q(z)$. (I) Let $P(z)$ and $Q(z)$ be real polynomials. If all the zeros of $P(z)$ lie in the half-plane $\Re(z) \leq a < b$ and if t is any real number such that $a < t < b$ and such that all the zeros of $Q(z)$ lie in the sector $|\arg(z - t)| \leq \pi/(p+1)$, then none of the first p derivatives of $f(z)$ can vanish at t. (II) Let $Q(z)$ be an arbitrary polynomial and G_p the region comprised of all points from which the ensemble of zeros of $Q(z)$ is seen from an angle of at least π/p. Then all the zeros of the pth derivative of $1/Q(z)$ lie in G_p. (III) Let $P(z)$ and $Q(z)$ have the degrees n and m respectively with $n \neq m$, let p and q be any real numbers with $p + q = n - m$ and let a, b, c be any complex numbers such that $f(a) f'(a) \neq 0$, ∞ and

(1) $p(a-b)^{-1} + q(a-c)^{-1} = f'(a)/f(a)$.

Then, if all the zeros of $f(z)$ lie in a region bounded by a circle C which passes through the points a, b and c, then at least one pole of $f(z)$ lies in this region or all the zeros and poles of $f(z)$ lie on C. Theorems (I) and (II) are proved by showing that the coefficient of z^n in the Taylor expansion for $f(t+z)$ does not vanish. Theorem III is proved by interpreting equation (1) as a centroid relation among the inverses in a of b, c and the zeros of $f(z)$.

Reviewer's note: While the proofs of (I) and (II) are novel, (I) and (III) are immediate consequences of a theorem of Bôcher and Walsh and (II) is in at least the

case $p=1$ contained in a theorem by the reviewer. [Cf. theorems (8, 2) and (20, 1) of the reviewer's book (second preceding review).]   *M. Marden* (Milwaukee, Wis.).

Sz. Nagy, Gyula. Über rationale Funktionen, deren Nullstellen und Pole an entgegengesetzten Seiten einer Geraden liegen. Hungarica Acta Math. 1, no. 4, 12–16 (1949).

The first set of theorems in this paper concern the polynomials

$$f(z) = (z-a_1)(z-a_2) \cdots (z-a_m),$$
$$g(z) = (z-b_1)(z-b_2) \cdots (z-b_n)$$

and the rational function $F(z) = f(z)/g(z)$. It is assumed that a line L through two given points α and β separates all the points a_j from all the points b_j, with $0 < \arg (\beta-a_1)/(\alpha-a_1) < \pi$, and that $0 < w = \arg F(\beta)/F(\alpha) \leq 2\pi$. Denoting by S the lens region comprised of the points from which the segment (α, β) subtends an angle of at least $w/(m+n)$, the author shows that either at least one zero of $f(z)g(z)$ lies in S or all these zeros lie on the boundary of S. His proof is based upon the variation in $\arg F(z)$ as z moves on L from α to β. The second set of theorems concern the zeros of the derivative $F'(z)$ of $F(z)$ in the case that all the a_j lie in the upper half-plane and each b_j is the conjugate imaginary $\bar{a}_j$ of the corresponding a_j. Denoting by H_k the equilateral hyperbola with vertices at a_k and $\bar{a}_k$, the author shows that no zero of $F'(z)$ lies interior or exterior to all the H_k. This follows from examination of the imaginary part of $F'(z)/F(z)$. A consequence of this result is that, if all the zeros of $f(z)$ lie exterior to an ellipse E with center on the real axis, then no zero of $F'(z)$ lies inside the circle concentric with E and tangent internally with E. [Reviewer's note: Instead of the phrase "no zero of $F'(z)$," the paper has the opposite phrase "every zero of $F'(z)$" in the two theorems stated above, but this appears to be a typographical error.]   *M. Marden.*

Angheluță, Th. On an upper limit for the moduli of the roots of an algebraic equation. Gaz. Mat., București **54**, 309–311 (1949). (Romanian)

The author gives an elementary proof of Walsh's theorem that all the zeros of the polynomial $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ lie in the circle $|z| \leq R_n = \sum_{j=1}^n |a_j|^{1/j}$. Let a sequence $P_k(z)$ of polynomials be defined by the recursion formula $P_k(z) = zP_{k-1}(z) - |a_k|$, for $k=1, 2, \cdots, n$, with $P_0(z) = 1$ and let $Q_k(z) = P_k(z+R_k)$. The author proves by induction on k that all the coefficients in each $Q_k(z)$ are positive. Hence, $|f(z)| \geq P_n(|z|) > 0$ for $|z| > R_n$. [Walsh's original proof was also elementary. Cf. ex. 9, p. 98, of the reviewer's book (fourth preceding review).]   *M. Marden.*

Hermann, A., et Souriau, J. M. Un critère de stabilité pour les équations caractéristiques à coefficients réels ou complexes. Recherche Aéronautique 1949, no. 9, 19–23 (1949).

The first part of this paper is an exposition on the use of Sturm sequences for calculating the number of zeros of a real polynomial on a given interval of the real axis. In the second part, the method is extended to the determination of the number of zeros of a real or complex polynomial to either side of the imaginary axis. [Reviewer's note. The material in the second part of this article is also not new. It is developed on pp. 127–132 of the reviewer's book (see the fifth preceding review).]   *M. Marden.*

Viola, Tullio. Sui determinanti di Hurwitz d'un'equazione algebrica, i cui coefficienti sono polinomi dipendenti da quanti si vogliono parametri reali. Boll. Un. Mat. Ital. (3) **4**, 40–45 (1949).

The author considers an algebraic equation whose coefficients are real algebraic functions of a (real) point P of an S_p and indicates some properties of the set of points P which ensure that the roots of the equation have negative real parts or are purely imaginary and simple.

C. Miranda (Naples).

Myasnikov, N. N. The criterion of Mihaĭlov and an estimate for the roots of the characteristic equation. Avtomatika i Telemehanika **10**, 267–273 (1949). (Russian)

Let $W(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ have all coefficients real. Mihaĭlov's criterion is that a necessary and sufficient condition for all roots of $W(z)$ to have negative real part is that $\Delta \arg W(i\omega) = -n\pi/2$ as ω varies from $+\infty$ to 0. This is an elementary deduction from a well-known theorem [Titchmarsh, The Theory of Functions, 2d ed., Oxford, 1939, p. 116]. The author shows how the roots of $W(z)$ may be approximated by plotting the curves $\lambda = $ constant and $\omega = $ constant of $W(i\omega - \lambda)$. The computations are given in complete detail for a sample polynomial of fourth degree.

A. W. Goodman (Lexington, Ky.).

Turán, Paul. On Descartes-Harriot's rule. Bull. Amer. Math. Soc. **55**, 797–800 (1949).

Let a real polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be expanded in terms of Laguerre polynomials $L_k(x)$ as $f(x) = b_0 L_0(x) + b_1 L_1(x) + \cdots + b_n L_n(x)$. Then, as Turán shows, the number of positive zeros of $f(x)$ is at least as great as the number of variations in the sequence $(c_0, c_1, \cdots, c_n)$ where $c_k = \sum_{j=0}^k (-1)^j C(k, j) b_j$ and $C(k, j)$ is the binomial coefficient. His proof follows from applying to the function $g(x) = e^{-x} f(x)$ the theorem of Fejér [C. R. Acad. Sci. Paris **158**, 1328–1331 (1914)] that the number of positive x at which a real function $g(x)$ changes sign on $0 < x < a \leq \infty$ is not less than the number of sign variations in the sequence of moments $M_k = \int_0^a g(t) t^k dt$, $k=0, 1, \cdots, n$. Turán's result supplements the theorem of Obreschkoff [Jber. Deutsch. Math. Verein **33**, 52–64 (1924)] that the number of zeros of $f(x)$ larger than the largest zero of $L_n(x)$ does not exceed the number of sign variations in the sequence $(b_0, b_1, \cdots, b_n)$. [To the author's reference list add M. Marden, Ann. of Math. (2) **33**, 118–124 (1932), where an independent derivation of Obreschkoff's theorem is given.]   *M. Marden.*

Tchakaloff, L. Sur le nombre des zéros non-réels d'une classe de fonctions entières. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 1, 9–12 (1949).

Let n be a positive integer; $p(t)$, a real or complex polynomial, and

$$P(x) = \int_{-1}^{1} [p(t)(x+it)^n + \bar{p}(t)(x-it)^n] dt.$$

The author shows that the nth degree real polynomial $P(x)$, if not identically zero, has at most m or $m-1$ nonreal zeros according as m is even or odd, and furthermore that, when not greater than n, these limits are attained by at least one $P(x)$ of degree n. When $n \leq m$, the theorem is obvious. When $n > m$, the author writes

$$P(x) = (x-i)^{n-1} Q(x) + [(x+i)^{n+1} - (x-i)^{n+1}] \sum_{k=0}^{m} C_k (x-i)^k,$$

where $Q(x) = \sum_{k=0}^{m} [C_k(x+i)^k - \bar{C}_k(x-i)^k]$, but where $C_m + \bar{C}_m = 0$. For any $x = -\cot \theta_r$, where $\theta_r = r\pi/(n+1)$ and

$r = 1, 2, \cdots, n$, $P(x_r) = (-1)^{n+1-r}Q(x_r)\csc^{n+1}\theta_r$. But there are at least $n-m$ intervals (x_r, x_{r+1}) which contain no zero of $Q(x)$ and for which either $P(x_r) = 0$ or $P(x_r)P(x_{r+1}) < 0$. Hence the first part of the theorem is established. To prove the second part of the theorem the author chooses the coefficients a_k in $p(t) = p_0(t) = \sum_{k=0}^{k} a_k t^{2k-2k}$ or $p(t) = t^{2k+1} + p_0(t)$ so that the first $h+1$ coefficients in the corresponding $P(x) = A_0 x^n + A_1 x^{n-2} + \cdots$ are positive. By Descartes' rule of signs, $P(x)$ has at least m or $m-1$ nonreal zeros according as m is even or odd and therefore by the first part of the theorem $P(x)$ has exactly these numbers of nonreal zeros. The theorem has an immediate extension to the function defined like $P(x)$ but with $(x \pm it)^n$ replaced by $e^{\pm its}$.

                M. Marden (Milwaukee, Wis.).

Karamata, J. Sur l'approximation de e^x par des fonctions rationnelles. Bull. Soc. Math. Phys. Serbie **1**, 7–19 (1949). (Serbian. Russian and French summaries)

Let p and q be positive integers and $a < b$. There exists exactly one polynomial $P(x)$ with

$$P(x) - P'(x) = (x-a)^p (x-b)^q.$$

Put $F(x) = P(x)e^{-x}$. Then $F(a) < F(b)$ or $F(a) > F(b)$ according as q is odd or even. If explicit formulas for $F(a)$ and $F(b)$ are introduced we obtain inequalities for e^t where $t = b - a$. For example, if $q = 1$, then

$$e^x \leq \sum_{\nu=0}^{p} \frac{x^\nu}{\nu!} + \frac{x^{p+1}}{(p+1)!} \frac{1}{1 - x/(p+1)},$$

provided $0 \leq x < p+1$. The case $p = q = n$ leads the author to the investigation of the polynomials

$$E_n(x) = \sum (2n-\nu)! x^\nu / (\nu!(n-\nu)!).$$

They have no real zero if n is even, and a single negative zero $-x_n$ if n is odd. The asymptotic behavior of $E_n(x)$ is given and it is shown that $E_n(x)/E_n(-x)$ leads to a continued fraction expansion for e^x.   *W. Feller.*

Madhava Rao, B. S., and Thiruvenkatachar, V. R. On an inequality concerning orthogonal polynomials. Proc. Indian Acad. Sci., Sect. A. **29**, 391–393 (1949).

Let $P_n(x)$ be Legendre's polynomial. The inequality $\Delta_n(x) = P_n^2 - P_{n-1}P_{n+1} \geq 0$ of Turán [cf. Szegő, Bull. Amer. Math. Soc. **54**, 401–405 (1948); these Rev. **9**, 429] is proved here by showing that $n(n+1)\Delta_n''(x) = -2(P_n')^2 \leq 0$. Similar arguments apply to the Hermite and Laguerre polynomials.
        G. Szegő (Stanford University, Calif.).

Pollaczek, Félix. Systèmes de polynomes biorthogonaux qui généralisent les polynomes ultrasphériques. C. R. Acad. Sci. Paris **228**, 1998–2000 (1949).

The author discussed recently [same vol., 1363–1365, 1553–1556 (1949); these Rev. **10**, 703] a generalization $P_n(x; a, b)$ of Legendre polynomials depending on the parameters a, b. He now extends this generalization in the direction of the ultraspherical polynomials. The system in question is orthogonal with the weight function

$$e^{(2t-\pi)\tau} \Gamma(\lambda + i\tau)\Gamma(\lambda - i\tau)(\sin t)^{2\lambda-1},$$

$$\tau = (a \cos t + b)/\sin t, \ a > |b|; \ 0 \leq t \leq \pi, \ \lambda > -\tfrac{1}{2}.$$
        G. Szegő (Stanford University, Calif.).

Videnskiĭ, V. S. On inequalities for the derivatives of polynomials. Doklady Akad. Nauk SSSR (N.S.) **67**, 777–780 (1949). (Russian)

Let $M_l(x)$ and $N_{l-1}(x)$ be real polynomials of the degrees indicated by their indices $(l \leq n)$, positive for $x > 1$, with

all their zeros lying on $[-1, 1]$ and separating each other. Write

$$T_{n-l}(x) = \cos[(n-l)\cos^{-1}x], \ S_{n-l}(x) = \sin[(n-l)\cos^{-1}x],$$

and determine $M_n(x)$ and $N_{n-1}(x)$ so that

$$M_n(x) \pm i(1-x^2)^{\frac{1}{2}} N_l(x)$$
$$= [T_{n-l}(x) + iS_{n-l}(x)][M_l(x) + i(1-x^2)^{\frac{1}{2}} N_{l-1}(x)].$$

Let $P_n(x)$ be a real polynomial of degree not exceeding n, such that on $[-1, 1]$

$$|P_n(x)|^2 \leq M_l^2(x) + (1-x^2)N_{l-1}^2(x)$$
$$= M_n^2(x) + (1-x^2)N_{n-1}^2(x) \equiv H(x).$$

The author shows that for $k \geq 1$

$$|P_n^{(k)}(x)| \leq \{[D^k M_n(x)]^2 + [D^k\{(1-x^2)N_{n-1}(x)\}]^2\}^{\frac{1}{2}},$$

where $D = d/dx$. The case $k = 1$ was given by S. Bernstein [C. R. Acad. Sci. Paris **190**, 237–240 (1930)] and the case $l = 0 (H(x) \equiv 1)$ by Schaeffer and Duffin [Bull. Amer. Math. Soc. **44**, 289–297 (1938)]. The author uses the method of Schaeffer and Duffin.   *R. P. Boas, Jr.* (Providence, R. I.).

Special Functions

✱**Magnus, Wilhelm.** Spezielle Funktionen der mathematischen Physik. Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 159–179. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

✱**Oberhettinger, Fritz, und Magnus, Wilhelm.** Anwendung der elliptischen Funktionen in Physik und Technik. Springer-Verlag, Berlin, 1949. vii+126 pp. 15.60 DM; bound, 18.30 DM.

Elliptic functions are not as fashionable as they used to be, and general courses in mathematical analysis do not contain, as a rule, more than the bare fundamentals of their theory. Yet in practical applications one needs the knowledge of a very large number of special formulas, and facility in handling them. The present book aims at introducing the reader to these formulas and illustrating various facets of the manipulative technique. The problems discussed have been selected with a view to giving the reader an all-round training in the mathematical technique, but they are grouped according to their nonmathematical subject-matter. The selection of topics for a book of this nature is necessarily a matter of personal taste, and a critical reader may have in mind one or two applications which he would have liked to see in the book; but there is hardly any subject treated that one would like to displace in order to make room for additional matter.

Chapter I gives a brief summary of the theory of elliptic functions. Elliptic integrals, theta functions, elliptic functions of both Jacobi and Weierstrass, and transformations are treated. There is an excellent collection of useful formulas. It is natural to the plan of the whole work that no proofs should be given in this chapter, but references to books where readers could find proofs and additional formulas would have been useful. Apparently, the authors did not wish to quote their well-known book [Formeln und Sätze für die speziellen Funktionen der mathematischen Physik, 2d ed., Springer-Verlag, Berlin, 1948; these Rev. **10**, 38] where the reader will find all that is required.

Chapter II is devoted to conformal representation and the closely related theory of Green functions in two-dimen-

sional potential theory. Explicit results are given for the rectangle and for the ellipse, and for the regions inside or outside a regular polygon, in particular for the regions inside a square or equilateral triangle. In chapter III the authors show typical applications to two-dimensional electrostatics. The fields of variously arranged conducting infinite strips are obtained, also fields of arrangements in which a strip is inside a cylindrical cavity of circular or square cross section. The field of a regular arrangement of parallel wires follows, with application to wires between two parallel plates or inside a cavity of rectangular cross section.

Chapter IV contains applications to problems of hydro- and aerodynamics. A brief discussion of vortices, single, or a regular pattern between two plane walls or in a channel of rectangular cross section, is followed by the theory of an infinitely long wing in a wind-tunnel or rectangular cross section, with an appendix on wind-tunnels of elliptic cross section. Chapter V is headed "miscellaneous examples," and it discusses the pendulum, the potential of a charged ellipsoid, the buckling of a column, and "Chebysheff" approximation.

The book is very well written, the exposition is lucid, the proofs are carefully arranged, with just sufficient detail to enable the reader to follow every step, and yet not so detailed as to make him lose patience. There are references to the literature in every chapter (except the first one). At the end of the book the authors have provided 14 pages of numerical tables of complete and incomplete elliptic integrals which will enable the reader to perform numerical computations to a modest degree of accuracy without having to take recourse to more elaborate books or tables. No reference is given in the book as to the origin of these tables; according to information received from the authors they are extracted from Legendre's tables, except for the table of the ratios of real and imaginary periods for which the source is Hayashi.   *A. Erdélyi* (Pasadena, Calif.).

✶**Lense, Josef. Reihenentwicklungen in der mathematischen Physik.** 2d ed. Walter de Gruyter & Co., Berlin, 1947. viii+226 pp. 18 DM.

The principal feature of this book [the first edition of which appeared in 1933] is a sound and rigorous presentation at a comparatively elementary level. A good course on advanced calculus, and the elements of complex variable theory (to the extent covered, say, in the first seven chapters of R. V. Churchill's recent book) is all the preparation needed for a thorough understanding of the author's exposition. For that matter, very little complex variable theory is needed outside the chapter on Bessel functions. From the point of view of the beginner, the almost complete absence of the phrase "it is easy to see" is particularly gratifying. All estimates needed in connection with series expansions, deformation of contours, etc., are carried out in detail: only routine algebra is left to the reader. It is inevitable, of course, that such an explicit presentation should involve some repetition and explanation of standard techniques, and an advanced student may find the going rather slow at times; but a beginner will be grateful for a moderate pace.

One might feel that the choice of the title is not very fortunate. This is not a book on the expansions occurring in mathematical physics, but rather a treatise on the properties of the special functions which enter into the expansions. Thus, for instance, there is a chapter of some

sixty odd pages on Legendre functions. Many properties of these functions are investigated, including the orthogonal property, and it is shown how to determine the coefficients in a uniformly convergent series of spherical harmonics; but there is no attempt made to discuss conditions under which an arbitrary function can be expanded in such a series.

After a brief introduction on polynomial approximation and Fourier series, chapter I deals with asymptotic series, Bernoulli polynomials, the Euler-Maclaurin expansion, and approximate evaluation of definite integrals. The notation for Bernoulli polynomials is not very clear, in that the same symbol $B_n(x)$ seems to stand sometimes for the polynomial, and at other times for the periodic function of period unity which coincides with the polynomial in the interval $(0, 1)$. Chapter II is on the gamma function. The asymptotic behaviour of the gamma function is investigated by means of the Euler-Maclaurin expansion. In addition to the more usual topics, the conformal mapping defined by the gamma function is also discussed. Chapter III is an exposition of the more elementary properties of orthogonal systems of functions. The process of orthogonalisation, Bessel's inequality, completeness, convergence in mean, Laguerre, Hermite, and Tchebysheff polynomials are the topics discussed in this chapter. Chapter IV, on Bessel functions, is the longest chapter in the book. Bessel's differential equation is introduced by considering the vibrations of a heavy chain and of a circular membrane, and conduction of heat in a sphere. The solutions of the differential equation are obtained as definite integrals of the Laplace type, and then evaluated in infinite series. Recurrence relations, more general integral representations, and Hankel's asymptotic expansions follow. Then come Sommerfeld's integrals, Airy integrals, and Debye's asymptotic expansions obtained by the method of steepest descent. Definite integrals containing Bessel functions are evaluated, the zeros of Bessel functions are investigated, and the conformal mapping defined by Bessel functions is briefly described.

The second largest chapter of the book is chapter V on Legendre functions. These are shown to arise when Laplace's partial differential equation is separated in spherical polar coordinates. Legendre polynomials are discussed first. Their generating function, the Fourier series representing them, recurrence relations, integral representations, zeros, asymptotic representations are obtained. Then the author turns to associated Legendre functions of integer degree and order. In addition to the topics already mentioned we have the orthogonal property, the expansion of x^n in a series of Legendre polynomials, the expansion of a plane wave in spherical waves, and the application of Legendre polynomials to mechanical quadrature. Legendre functions of the second kind are given slightly less space. Lastly, spherical surface harmonics, their orthogonal property, their addition theorem, and their application to the solution of boundary value problems of Laplace's equation are discussed. A kind of postscript to this chapter gives the expansion of a spherical wave whose centre does not coincide with the origin of the coordinate system. Chapter VI is devoted to ellipsoidal harmonics. Laplace's equation is separated in ellipsoidal coordinates, and the various types of Lamé polynomials are established. In order to avoid elliptic functions, the author refrains from the introduction of uniformising variables. Relations of ellipsoidal surface harmonics with spherical surface harmonics, and the degenerate case of confocal quadrics of revolution are discussed

briefly, but there is no adequate treatment of spheroidal harmonics. Theorems on the zeros of Lamé polynomials, the orthogonal property, and the application of Lamé polynomials to boundary value problems of Laplace's equation close this last chapter of the book.

The book is an excellent introduction to the special functions of mathematical physics. It is not only explicit, but also very precise. Lapses such as on p. 129 where it is stated that all derivatives of $1/r$ are solutions of Laplace's equation (where the author means all partial derivatives with respect to Cartesian coordinates, or indeed in any fixed direction) are very rare. Historical remarks and references to the originators of particularly neat and attractive proofs will put the whole subject in perspective for the student, and will teach him to value elegance as much as thoroughness in exposition and precision in the statement of results.

A. *Erdélyi* (Pasadena, Calif.).

✳Lense, Josef. **Reihenentwicklungen der mathematischen Physik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 181–188. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Rosenberg, R. L. **The loss of energy of slow negative mesons in matter.** Philos. Mag. (7) **40**, 759–769 (1949).

The main part of the paper is of interest only to physicists. In the appendix, the double integral

$$\int_0^\pi \int_0^\infty e^{-\alpha r+\beta r_2}(R-r\cos\theta)r_2^{-3}(kr)^{-\frac{1}{2}}J_{l+\frac{1}{2}}(kr)$$
$$\times P_l(\cos\theta)r^2\sin\theta\,d\theta\,dr$$

is reduced to a sum of simple integrals. Here r, θ, φ are spherical polar coordinates, r_2 is the distance of (r, θ, φ) from $(R, 0, 0)$; α, β, k, R are constants, and l is a nonnegative integer; α and β depend on R. For large R (when β is very small) the integral is approximately $4\alpha^2\frac{1}{2}\pi^{-\frac{1}{2}}(\alpha^2+k^2)^{-2}R^{-2}$.

A. *Erdélyi* (Pasadena, Calif.).

Nordon, Jean. **Sur diverses équations différentielles liées aux fonctions de Lord Kelvin.** Bull. Sci. Math. (2) **73**, 37–47 (1949).

The functions are defined by

(1) $\text{ber}_\lambda x+\text{bei}_\lambda x=e^{\frac{1}{2}\lambda\pi i}I_\lambda(xe^{\frac{1}{4}\pi i})$,
             $\text{ker}_\lambda x+\text{kei}_\lambda x=e^{\frac{1}{2}\lambda\pi i}K_\lambda(xe^{\frac{1}{4}\pi i})$

and

(2) $B_\lambda x=I_\lambda(2x^{\frac{1}{2}}e^{\frac{1}{4}\pi i})I_\lambda(2x^{\frac{1}{2}}e^{-\frac{1}{4}\pi i})$,
             $K_\lambda x=K_\lambda(2x^{\frac{1}{2}}e^{\frac{1}{4}\pi i})K_\lambda(2x^{\frac{1}{2}}e^{-\frac{1}{4}\pi i})$,
             $M_\lambda x+iN_\lambda x=I_\lambda(2x^{\frac{1}{2}}e^{\frac{1}{4}\pi i})K_\lambda(2x^{\frac{1}{2}}e^{-\frac{1}{4}\pi i})$.

The author gives the fourth order differential equation satisfied by the functions (1), a third order equation satisfied by a combination of functions (2), and the power series expansions of the functions (2); he discusses also the differential equation satisfied by the indefinite integrals of the functions (2). Lastly, he gives integral representations and connections with generalised hypergeometric functions.

A. *Erdélyi* (Pasadena, Calif.).

Schmid, Hermann Ludwig. **Störungsrechnung bei dreigliedrigen Rekursionen.** I. Math. Nachr. **1**, 377–398 (1948).
Schmid, Hermann Ludwig. **Störungsrechnung bei dreigliedrigen Rekursionen.** II. Math. Nachr. **2**, 35–44 (1949).

The coefficients of all the known expansions of Mathieu functions, spheroidal wave functions, Lamé functions, and related functions satisfy three-term recurrence relations which depend on two parameters: one is a multiple of the "wave number" and may be considered a given quantity in many problems, while the other is a separation parameter capable of certain characteristic values which will produce a solution of the required nature. The author investigates a class of three-term recurrence relations which includes as particular cases all the above mentioned systems and many others.

The coefficients $a_{2k}^{(r)}$ satisfy

(1) $p_{r+2k-1}a_{2k-2}^{(r)}+q_{r+2k}a_{2k}^{(r)}+r_{r+2k+1}a_{2k+2}^{(r)}=0,$

where k is any integer, p, q, r are polynomials in the "perturbation parameter" γ^2 (given), and q contains a second parameter λ (not given). We are interested in nontrivial solutions of (1) for which (2) $p_{r-2k+1}a_{-2k}^{(r)}$ and $r_{r+2k-1}a_{2k}^{(r)}$ are divisible by γ^{2k} for all positive integers k. The assumptions on the coefficients p, q, r are as follows:

(3) $p_{r+2k-1}r_{r+2k-1}=\gamma^{2m}l_{r+2k-1}$

for all integers k, where m is a fixed positive integer and t is independent of γ^2; (4) $q_{r+2k}=\lambda-q_{r+2k}^*$ for all integers k, where q^* is independent of λ and is a polynomial of degree at most $m-1$ in γ^2; and (5) $q_r^*-q_{r+2k}^*$ is not divisible by γ^2 for all nonzero integers k.

In the first paper the author proves the existence of characteristic values of $\lambda^{(r)}$ for the problem (1), (2), expands these characteristic values in powers of γ^2, obtains recurrence relations for the computation of the coefficients of this power series expansion, proves the uniqueness (up to a constant factor) of the solution of (1), (2) belonging to a characteristic value of λ, and discusses briefly the modifications which become necessary when the system (1) does not extend over all values of k, but terminates in one direction while it extends indefinitely in the other.

The most important result of the second paper is a much more elegant representation of the coefficients in the power series expansion of the coefficients $\lambda_{2r}^{(r)}$. There are many other results which cannot be summarised briefly.

A. *Erdélyi* (Pasadena, Calif.).

Harmonic Functions, Potential Theory

✳Maruhn, Karl. **Potentialtheorie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 11–19. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This summary contains a brief account of papers on (A) special solutions of the potential equation, (B) boundary value problems, (C) connections with boundary value problems for other differential equations, (D) connections with the theory of functions, (E) biharmonic functions. Most of the material is now generally available. An exception is a contribution in a manuscript by H. Schubert on a nonlinear boundary value problem for the circle. This is of a type similar to one of the nonlinear problems studied by K. Maruhn [Math. Z. **51**, 36–60 (1947); these Rev. **9**, 433], but, unlike Maruhn's problem, admits a finite number of discontinuities in certain functions used in formulating the boundary condition.   F. W. *Perkins* (Hanover, N. H.).

Uspenskiĭ, V. A. Geometrical deduction of the fundamental properties of harmonic functions. Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 201–205 (1949). (Russian)

L'auteur retrouve quelques propriétés fondamentales des fonctions harmoniques dans le plan à partir de l'égalité de la moyenne, en présentant la moyenne de la manière suivante, équivalente dans le cas de fonctions intégrables au sens de Riemann: ce sera pour chaque circonférence C une fonctionnelle linéaire croissante de fonction réelle sur C, égale à 1 si la fonction vaut 1 et invariante par translation et symétrie. Soulignons une nouvelle solution de problème de Dirichlet dans le plan: on se ramène au cas d'une donnée étagée, puis d'une donnée constante sur les deux arcs séparés par une corde AB. Solution à l'aide de la fonction égale à l'angle AMB. *M. Brelot* (Grenoble).

Brelot, Marcel, et Choquet, Gustave. Lignes de Green et mesure harmonique. C. R. Acad. Sci. Paris **228**, 1556–1557 (1949).

It is known, for a simply connected domain D, that the orthogonal trajectories of the "circles of Green" C_λ of the equation $G(O, M) = \lambda$ (O fixed in D) determine a one-to-one correspondence between the prime ends of D and the points of C_λ; it is also known that this correspondence preserves (in a certain sense) the harmonic measure, and that almost all the orthogonal trajectories have finite length. The authors extend these results to arbitrary domains in n-space, $n \geq 2$. They obtain, for example, the result that the set of finite frontier points (of a domain) that are not finitely accessible constitutes a set of harmonic measure zero. The authors' technique is to introduce a measure in the space of regular "Green lines." *M. Reade.*

Brelot, Marcel. Le problème de Dirichlet géodésique. C. R. Acad. Sci. Paris **228**, 1790–1792 (1949).

Let Ω denote a bounded domain in Euclidean n-space, $n \geq 2$, and consider the geodesic metric; that is, the distance between two points of Ω is the greatest lower bound of the lengths of the Jordan arcs in Ω joining those points. The completion of the metric space Ω yields a space Ω^*; the set $\Omega^* - \Omega$ will be called the geodesic frontier F of Ω. The author then considers Dirichlet problems for Ω and F, called geodesic problems, in terms of the geodesic metric; he obtains a necessary and sufficient condition that the geodesic problem have a solution for real-valued f defined on F. The present outline utilizes a technique developed earlier [Brelot and Choquet, paper reviewed above; Brelot, Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 167–200 (1946); these Rev. **8**, 581]. *M. Reade.*

Pescarini, Angelo. Su alcuni sistemi di equazioni lineari alle derivate parziali. Boll. Un. Mat. Ital. (3) **4**, 63–67 (1949).

L'auteur considère le système (1) $\sum_{j=1}^n \Delta_{ij} X^j = 0$, où $\Delta_{ij} = \sum_{k=1}^n \alpha_{kij} \partial/\partial x_k$ (α_{kij} constantes) et introduit les polynomes caractéristiques $\Delta_{ij}^* = \sum_{k=1}^n \alpha_{kij} \xi_k$. On suppose que le déterminant $D = |\Delta_{ij}^*|$ est $\neq 0$. Alors pour que les solutions de (1) soient harmoniques, il faut et suffit que D soit, à un facteur près constant $\neq 0$, égal à $(x_1^2 + \cdots + x_m^2)^{n/2}$ et que le mineur D_n de D soit divisible par $(x_1^2 + \cdots + x_m^2)^{k-1}$. Il est donc nécessaire que n soit pair. *M. Brelot.*

Berker, Ratip. Sur certaines propriétés du rotationnel d'un champ vectoriel qui est nul sur la frontière de son domaine de définition. C. R. Acad. Sci. Paris **228**, 1630–1632 (1949).

This paper quotes several theorems obtained by the author in generalizing to three-space the plane problem treated by Kampé de Fériet [Math. Mag. **21**, 74–79 (1947); these Rev. **9**, 433]. No proofs are given. An example of the results obtained is the following. Let **V** denote the velocity vector of an incompressible fluid occupying a finite space domain Δ bounded by a surface S and satisfying certain regularity conditions in this closed domain. Further, let **a** denote a vector field which is harmonic and solenoidal in $\Delta + S$ and satisfies certain other regularity conditions in $\Delta + S$. Then, the vorticity vector **w** is orthogonal to **a** in Δ. That is, $\int_\Delta \mathbf{a} \cdot \mathbf{w} d\tau = 0$. *N. Coburn* (Ann Arbor, Mich.).

Laporte, Otto. Polyhedral harmonics. Z. Naturforschung **3a**, 447–456 (1948).

The author discusses the problem of determining regions bounded by planes that meet in one point for which it is possible to find simple solutions of the wave equation. Introducing spatial polar coordinates, he seeks solutions of the three-dimensional Laplace equation which vanish, or whose normal derivatives vanish, on planes meeting in one point. Rules are given for the construction of homogeneous harmonic polynomials which are invariant under the substitutions of the four polyhedral groups; these form two sets according to whether they vanish on the planes of symmetry (odd harmonics), or whether their normal derivatives vanish on these planes (even harmonics). Two methods for obtaining these polyhedral harmonics are given: one method depends on the observation that the Laplace operator applied to a primitive homogeneous polynomial invariant of a polyhedral group must yield a linear combination of primitive invariants of degree two less; the other is an adaption of Maxwell's method of poles.
E. F. Beckenbach (Los Angeles, Calif.).

Verblunsky, S. Inequalities for the integrals of positive harmonic functions along contours. J. London Math. Soc. **24**, 149–153 (1949).

This article generalizes previous results of Gabriel [same J. **21**, 87–90 (1946); these Rev. **8**, 461] and Reuter [same J. **23**, 56–58 (1948); these Rev. **10**, 39]. Let C be any closed convex surface in the Euclidian n-dimensional space, I the interior volume, u any nonnegative harmonic function in I, continuous in $I + C$, Γ a sphere of radius R which is contained in I. (1) If through each point P of C we can draw a sphere C_P of radius r which is exterior to I, then $\int_\Gamma u dS \leq (2 + R/r) \int_C u dS$. (2) If through each point P of C we can draw a sphere γ_P of radius r which is contained in $I + C$ and contains Γ; if d denotes the lower bound of the distance between γ_P and Γ for $P \epsilon S$, then

$$\int_\Gamma u dS \geq (R+d) r^{-1} \{R/[2r - (R+d)]\}^{n-1} \int_C u dS.$$

These inequalities are easily deduced from a majoration and a minoration of the harmonic measure (expressed by the normal derivative of the Green's function) by replacing successively C by C_P and γ_P. The problem is treated by the author only for $n = 2$. *L. Schwartz* (Nancy).

Gabriel, R. M. Some inequalities concerning subharmonic functions. J. London Math. Soc. 24, 154–156 (1949).

This article follows a previous one of the same author [same J. 21, 87–90 (1946); these Rev. 8, 461] and another of Reuter [same J. 23, 56–58 (1948); these Rev. 10, 39]. The author remarks that the previous results are immediate consequences of the following statement: if S is any closed convex surface, V the interior volume, then the harmonic measure μ_x over S with respect to $x \varepsilon V$ is majorized by $(2/k)\Omega_x$, where Ω_x is the solid angle over S with respect to x, and k the surface of the unit sphere. This property is known and was already pointed out by the reviewer in the review of Reuter's paper [Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 68], but the author's proof is very short. It can be written as follows: $\mu_x - (2/k)\Omega_x$ is a harmonic function of x (with values in the space of the measures over S): it becomes trivially nonpositive when x approaches S, so it is nonpositive for every $x \varepsilon V + S$. [At the beginning of the paper, read $U(Q)$ instead of $|U(Q)|$.]

L. Schwartz (Nancy).

Minakshisundaram, S., and Pleijel, Å. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. Canadian J. Math. 1, 242–256 (1949).

This paper offers an extension of a method of Carleman's from a Euclidean to a curvilinear setup, and the results of Carleman have been both generalized and added to; an important tool of investigation is Hilbert's parametrix. If a Laplacian $\Delta\varphi$ on a Riemannian manifold is endowed with a boundary condition that will make it the inverse of a totally continuous operator in Hilbert space, except perhaps for an eigenvalue $\lambda_0 = 0$ with the eigenfunction $\varphi_0(P) = $ constant in the case of a compact manifold "without boundary," then the series $\sum_m \lambda_m^{-s} \varphi_m(P) \varphi_m(Q)$ is an entire function of the complex variable s having zeros at nonpositive integers; except in the case $P = Q$, in which case it may have simple poles as certain integers or half-integers. Also, in the case of a compact manifold, the series $\sum_{m=1}^{\infty} \lambda_m^{-s}$, without the eigenfunctions $\varphi_m(P)$, is likewise meromorphic in the manner just stated. Finally, a Tauberian argument gives

$$\sum_{\lambda_m \leq T} \varphi_m^2(P) \sim T^{n/2}(2\pi^{\frac{1}{2}})^{-n}/\Gamma(\tfrac{1}{2}n + 1)$$

in the noncompact case and also

$$\sum_{\lambda_m \leq T} 1 \sim VT^{n/2}(2\pi^{\frac{1}{2}})^{-n}/\Gamma(\tfrac{1}{2}n + 1)$$

in the compact case. *S. Bochner* (Princeton, N. J.).

Minakshisundaram, S. Zeta functions on the sphere. J. Indian Math. Soc. (N.S.) 13, 41–48 (1949).

The properties of the Laplacian for a compact manifold which were obtained in the paper reviewed above are derived for an n-dimensional sphere from special properties of Bessel functions and Gegenbauer polynomials.

S. Bochner (Princeton, N. J.).

Bader, Roger. La théorie du potentiel sur une surface de Riemann. C. R. Acad. Sci. Paris 228, 2001–2002 (1949).

Extension of known potential-theoretic results to Riemann surfaces. The notion of capacity of the ideal boundary is introduced and is related to the classification of boundaries (null and positive) due to R. Nevanlinna.

M. Heins (Providence, R. I.).

Kodaira, Kunihiko. Harmonic fields in Riemannian manifolds (generalized potential theory). Ann. of Math. (2) 50, 587–665 (1949).

This paper gives a complete account of the author's work on the theory of harmonic integrals. Nearly all of it has been published before [Proc. Imp. Acad. Tokyo 20, 186–198, 257–261, 353–358 (1944); Math. Japonicae 1, 6–23 (1948); these Rev. 7, 329; 10, 211], but some details omitted from earlier papers have been included, and the subject is taken further by the consideration of integrals of the second and third kinds. Moreover, for completeness, some topological preliminaries, and proofs of de Rham's theorems [J. Math. Pures Appl. (9) 10, 115–200 (1931)] are given, the proof of the first being essentially de Rham's. The account given is a complete account of the work done on the theory of harmonic integrals by means of Weyl's method of orthogonal projections. But, as the author explains in a postscript, he did not have access to, and was hence unaware of, other work on the theory until his attention was drawn to it by the referee. Thus there is no reference to the work of de Rham [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 22, 135–152 (1946); these Rev. 8, 603], Bidal and de Rham [Comment. Math. Helv. 19, 1–49 (1946); these Rev. 8, 93], or Weil [Comment. Math. Helv. 20, 110–116 (1947); these Rev. 9, 65] or the later work of the reviewer.

W. V. D. Hodge (Cambridge, Mass.).

Andreyev, B. A. Calculations of the spatial distribution of potential fields and their application to prospecting geophysics. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 11, 79–92 (1947). (Russian. English summary)

The value U_P at a point $P(0, 0, -h)$ in the half-space $z < 0$ of a harmonic function $U(x, y, z)$ known in the plane $z = 0$ is expressed by the classical formula

$$2\pi U_P/h = \int\int r^{-3} U(x, y, 0) dS$$

taken over the whole infinite plane $z = 0$. Considering the particular two-dimensional case of this formula, when U does not depend on y and $\pi U_P/h = \int_{-\infty}^{\infty} U(x, 0)(x^2 + h^2)^{-1} dx$, the author discusses the application of this simple integral to various two-dimensional problems of gravimetric and magnetic surveying. [His important formula (8) is wrong and its numerator $\Delta U^*(l_m)$ must be replaced by the sum $h \cdot \Delta U^*(l_m) + h \cdot \bar{U}(0)$. The coefficient 4 in (10), if corrected, becomes 1.] *E. Kogbetliantz* (New York, N. Y.).

Andreev, B. A. Calculations of the spatial distribution of potential fields and their application to prospecting geophysics. II. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. 13, 256–267 (1949). (Russian)

The values $U(x, h)$ of a harmonic function $U(x, z)$ on a parallel $z = h > 0$ to the OX-axis $z = 0$ can be computed, if its values $U(x, 0)$ on the OX-axis are known. They verify the integral equation

(E) $$\pi U(x, 0) = h\int_{-\infty}^{\infty} U(s, h)[h^2 + (s-x)^2]^{-1} ds.$$

Solving (E) by successive approximations

$$\pi U_{n+1}(x, h) = \pi U(x, 0) + \pi U_n(x, h)$$
$$- h\int_{-\infty}^{\infty} U_n(s, h)[h^2 + (s-x)^2]^{-1} ds,$$

the author proves their uniform convergence under the con-

dition $h < z_{min}$, where z_{min} denotes the greatest lower bound of the depth of sources. The application of this method to problems of applied geophysics is not considered.

E. Kogbetliantz (New York, N. Y.).

Differential Equations

✱Horn, J. **Gewöhnliche Differentialgleichungen.** 5th ed. Walter de Gruyter & Co., Berlin, 1948. viii+237 pp. 16 DM.

This is a new and somewhat revised edition of a well-known and deservedly popular textbook. It presents the usual parts of the theory of ordinary differential equations clearly and rigorously, in a way that is well suited to the needs of graduate students. It seems to the reviewer that the book should appeal especially to readers who are interested in the subject primarily for the sake of its applications in physics and engineering. The chief differences between this edition and the preceding ones are to be found in the more ample treatments of the hypergeometric functions and of the representation of solutions by asymptotic series.

L. A. MacColl (New York, N. Y.).

Sarantopoulos, Spyridon B. **Generalization of some linear differential equations of mathematical physics.** Prakt. Akad. Athēnōn **20** (1945), 390–394 (1949). (English. Greek summary)

The author states the linear differential equation which can be solved by the contour integral

$$u(z) = \sigma(z) \int_C e^{\varphi(t)} [f(t)]^{\eta}(t-z)^{-\eta-1} dt,$$

where

$$\varphi(t) = -(\mu\eta+1) \int_{t_0}^{t} \frac{g(u)}{f(u)} du,$$

$f(z)$ and $g(z)$ are polynomials of degrees $\mu+1$ and μ respectively, with leading coefficients unity, η is an arbitrary number, and $\sigma(z)$ an arbitrary function. He names several particular cases of this equation, and gives the modification necessary when $\mu+\eta=0$. *A. Erdélyi* (Pasadena, Calif.).

Gran Olsson, R. **Rigorous solution of a differential equation in soil mechanics.** Quart. Appl. Math. 7, 338–342 (1949).

An examination of the indeterminate form assumed at the initial value by the explicit solution of a specific linear equation of the first order. *P. Franklin* (Cambridge, Mass.).

Péyovitch, T. **Sur les transformations de certaines équations différentielles.** Bull. Soc. Math. Phys. Serbie 1, 41–44 (1949). (Serbian. Russian and French summaries)

The differential equation $y' = f(x, y)$ is changed by the transformation $y = uz$. The author discusses the conditions under which it is possible to choose $u(x)$ so that the variables in the new equation can be separated. A similar problem is treated for systems of equations. *W. Feller*.

Karanikoloff, Chr. **Sur une équation différentielle considérée par Kummer.** C. R. Acad. Bulgare Sci. Math. Nat. 2, no. 1, 25–28 (1949).

The author finds power series solutions of the equation $d^n y/dx^n = x^m y$ and discusses their singularities at $x = 0$.

P. Franklin (Cambridge, Mass.).

Leighton, Walter. **A substitute for the Picone formula.** Bull. Amer. Math. Soc. 55, 325–328 (1949).

The following theorem is proved. If the equation $(ru')' + pu = 0$ has a solution with zeros at x_1 and x_2, and if $r(x) > r_1(x) > 0$ on $x_1 \leq x \leq x_2$, then every solution of the equation $(r_1 u')' + p_1 u = 0$ has a zero on (x_1, x_2) if the solution $w(x)$ of the system $[(r-r_1)w']' + (p-p_1)w = 0$, $w(x_1) = 0$, is nonvanishing on (x_1, x_2). The proof depends on a lemma from the calculus of variations. *R. E. Langer*.

Wintner, Aurel. **On the smallness of isolated eigenfunctions.** Amer. J. Math. 71, 603–611 (1949).

L'autore considera l'equazione (1) $x'' + (f(t) + \lambda)x = 0$ nell'ipotesi che sia $\lim \sup_{t \to \infty} f(t) < +\infty$ e dimostra che, se λ_0 è un punto isolato dello spettro della (1) ed $x = y(t)$ è una soluzione di quadrato sommabile in $(0, \infty)$ della (1) per $\lambda = \lambda_0$, si ha $y(t) = O(t^{-N})$ per ogni N. *C. Miranda*.

Hartman, Philip, and Wintner, Aurel. **Oscillatory and nonoscillatory linear differential equations.** Amer. J. Math. 71, 627–649 (1949).

Gli autori studiano il comportamento asintotico degli integrali dell'equazione (1) $x'' + f(t)x = 0$, nell'ipotesi (2) $0 \leq \lim \inf_{t \to \infty} f(t) \leq \infty$ esaminando i vari casi che possono presentarsi. Dimostrano fra l'altro l'esistenza di una successione d'intervalli (t_n', t_n'') che tendono ad ∞ per $n \to \infty$, sui quali ogni integrale della (1) si mantiene limitato. Uno studio analogo è svolto sostituendo l'ipotesi (2) con l'altra che la (1) sia di tipo non oscillatorio. I risultati ottenuti, nell'ipotesi che la (1) contenga un parametro λ $(f = g + \lambda)$ conducono a stabilire varie proprietà dello spettro hilbertiano della (1). *C. Miranda* (Napoli).

Hartman, Philip, and Wintner, Aurel. **A separation theorem for continuous spectra.** Amer. J. Math. 71, 650–662 (1949).

In (1) $(px')' + (q + \lambda)x = 0$ let $p = p(t) > 0$, $q = q(t)$ be real and continuous for $0 \leq t < \infty$, and let λ be a real parameter. It is assumed that not all solutions of (1) are of class (L^2) on the half-line $0 \leq t < \infty$. Then the conditions (2a) $\int_0^\infty x^2(t) dt = 1$ and (2b) $x(0) \sin \alpha - p(0)x'(0) \cos \alpha = 0$ determine an eigenvalue problem for (1). The paper studies the dependence of this eigenvalue problem on the parameter α. The investigation is complicated by the possible presence of a continuous spectrum. The main result is the following separation theorem. If, for a given α, the interval between two isolated eigenvalues does not contain any points of the spectrum corresponding to this value of α, then it contains exactly one point of the spectrum belonging to every other value of α (mod π).

In an appendix some facts are proved concerning the continuous spectrum and the closure of the point spectrum. In particular, let S^α be the interior of the set of λ-values for which (1) has solutions satisfying (2a); then (a), the set S^α contains no point of the continuous spectrum and (b), the closure of the point spectrum is nowhere dense on S^α. These statements are valid for all values of α. The proof is based on an expression for the (discrete and continuous) Fourier coefficients and the generalized Parseval identity.

W. Wasow (Los Angeles, Calif.).

Gusarov, L. A. **On the boundedness of the solutions of a linear equation of the second order.** Doklady Akad. Nauk SSSR (N.S.) 68, 217–220 (1949). (Russian)

The author shows that all solutions of $y'' + p(x)y = 0$ are bounded for $x \geq x_0$ if $b^2 \geq p(x) \geq a^2 > 0$, and $p'(x)$ is continuous and of bounded variation over (x_0, ∞). *R. Bellman*.

El'śin, M. I. Qualitative problems on the linear differential equation of the second order. Doklady Akad. Nauk SSSR (N.S.) **68**, 221–224 (1949). (Russian)

The author discusses the problem of finding a representation of the solutions of $x'' + p(t)x' + q(t)x = 0$ in the form $x = a(t) \cos (b(t) + c)$ by the use of suitable transformations of variable. He then points out the application of this result in determining the oscillatory properties of the solutions.

R. Bellman (Stanford University, Calif.).

Krasnuškin, P. E. On the asymptotic representation of solutions of the wave equation. Vestnik Moskov. Univ. **3**, no. 6, 73–76 (1948). (Russian)

In the equation $d^2\phi/dz^2 + W(z)\phi = 0$ let $W(z)$ be an analytic function of z in which z occurs only in the combination $\epsilon^{\frac{1}{2}}z$, where ϵ is a small positive parameter. In order to solve (1) asymptotically for small ϵ the author transforms (1) into a simpler equation $d^2\psi/d\zeta^2 + P(\zeta)\psi = 0$ by setting $\phi = f^{\frac{1}{2}}\psi$, $dz/d\zeta = f$. The problem of finding $f(z)$ when $P(\zeta)$ is prescribed is solved formally by writing f^2 and ζ as power series in ϵ with indeterminate coefficients that are functions of $\epsilon^{\frac{1}{2}}z$. Then ϕ is calculated approximately by using only the first terms of these series. No complete discussion of the validity of this procedure is given, but conditions are formulated which guarantee the smallness of the second term in the resulting expression for ϕ. A physical interpretation in terms of wave propagation in nonhomogeneous media is added.

W. Wasow (Los Angeles, Calif.).

Yakubovič, V. A. A certain criterion for reducibility of a system of differential equations. Doklady Akad. Nauk SSSR (N.S.) **66**, 577–580 (1949). (Russian)

If $A(t)$ is an n by n matrix of continuous (complex-valued) functions of t, the system of linear differential equations (*) $dx/dt = A(t)x$ is said to be reducible if there exists a matrix $P(t)$, $\|P(t)\| \leqq$ constant, $\|P(t)^{-1}\| \leqq$ constant $(\|P\| = \{\sum_{i,j} |p_{ij}|^2\}^{\frac{1}{2}}, P = (p_{ij}))$, which transforms equation (*) by the substitution $x = P(t)y$ into $dy/dt = Ky$, where K is a constant matrix. It is known, in the case of a periodic system $A(t+w) = A(t)$, that equation (*) is reducible. The author states several generalizations of this result and proves one, namely: the equation (*) is reducible if $A(t+w) = SA(t)S^{-1}$, where S is a matrix equivalent to a unitary matrix. Another criterion for reducibility is obtained by applying a theorem previously proved by the author [same Doklady (N.S.) **63**, 363–366 (1948); these Rev. **10**, 535]: if $\int_{t_0}^{\infty} t^{m-2}\|A(t) - K\| dt$ converges, where m is the dimension of the largest box in the canonical form of the matrix K, then (*) possesses a fundamental matrix $P(t)e^{Kt}$, $P(t) \to E$ (the unit matrix) as $t \to \infty$, i.e., is reducible as $t \to \infty$. As a corollary the author obtains the result: if R is a constant matrix, $G(t)$ continuous for $0 < t \leqq t_0$, and $G(t) = O(t^{-\alpha})$, $t \to +0$, $\alpha < 1$, then the system $dx/dt = (R/t)x + G(t)x$ possesses a fundamental matrix of the form $P(t)t^R$, where $P(t) \to E$, $t \to +0$. As the author states, this was proved by A. Wintner for the special case when $\alpha = 0$ and the "cross-modulus" of R satisfies $\sup_{\|x\|=1} \|XR - RX\| < 1$ [Amer. J. Math. **68**, 185–213 (1946); these Rev. **8**, 71].

E. A. Coddington (Cambridge, Mass.).

Lur'e, A. I. On a canonical form of the equations of the theory of automatic regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 651–666 (1948). (Russian)

The system

$$\dot{y}_k = \sum_{l=1}^{n} b_{kl}y_l + h_k f(s), \qquad k = 1, \cdots, n,$$

$$s = \sum_{l=1}^{n} j c_l y_l$$

is put into the canonical form

$$\dot{x}_i = \lambda_i x_i + f(s), \qquad i = 1, \cdots, n,$$

$$\dot{s} = \sum_{l=1}^{n} \beta_l x_l - r f(s)$$

by means of a linear transformation $x_i = \sum_{l=1}^{n} c_i{}^l y_l$, in the case where the matrix $B = (b_{kl})$ has simple characteristic roots $\lambda_1, \cdots, \lambda_n$. Formulas are given for the constants $c_i{}^l$, β_l, r. Some physical examples are considered.

J. G. Wendel (New Haven, Conn.).

LaSalle, J. Uniqueness theorems and successive approximations. Ann. of Math. (2) **50**, 722–730 (1949).

The author considers the system

$$y_i' = f_i(x, y_1, \cdots, y_n) = f_i(x, y)$$

in one independent and n dependent variables. Let G be the class of functions $g(x)$ continuous on $0 \leqq x \leqq a$ and vanishing for $x = 0$. Let the f's be continuous in a region defined by $0 \leqq x \leqq a$, $l_i(x) \leqq y_i \leqq u_i(x)$, where $l_i(x)$, $u_i(x)$ belong to G. An approximating sequence $T^m(g(x))$ is generated by replacing y by a vector $g(x)$ with components from G and integrating $f(\sigma, g(\sigma))$. If T^m converges uniformly to a solution, $g(x)$ is called a zero approximation. If the f's are nondecreasing functions of the y's and if $l_i'(x) \leqq f_i[x, l(x)]$, $f_i[x, u(x)] \leqq u_i'(x)$, then $l(x)$ and $u(x)$ are zero approximations. For the case where the f's satisfy a generalized Lipschitz condition of considerable complexity the paper proves a uniqueness theorem which includes those of Osgood, Montel and Nagumo and also a theorem which specifies a subset of G composed of zero approximations and generalizes a theorem of Wintner [Amer. J. Math. **68**, 13–19 (1946); these Rev. **7**, 297]. For the case of more than one solution a result which leads to bounds on the difference of solutions is given. *J. M. Thomas* (Durham, N. C.).

Al'muhamedov, M. I. On conditions for the existence of stable and unstable centers. Doklady Akad. Nauk SSSR (N.S.) **67**, 961–964 (1949). (Russian)

This paper contains various necessary and sufficient conditions for a center for the differential equation in polar coordinates

$$(1) \qquad \frac{d\rho}{d\omega} = \frac{\rho^2\varphi_1 + \rho^3\varphi_2 + \cdots + \rho^{n+1}\varphi_n}{1 + \rho\psi_1 + \rho^2\psi_2 + \cdots + \rho^n\psi_n},$$

where ρ_i, ψ_i are polynomials in $\sin \omega$, $\cos \omega$. In particular one may reduce to (1) the equation

$$(2) \qquad dy/dx = -\{x + P_n(x, y)\}/\{y + Q_n(x, y)\},$$

where P_n, Q_n are polynomials of degree n beginning with terms of degree at least two. It is well known that for (1) the origin is a focus or a center. To distinguish between them the author endeavors to find a closed curve γ, (3) $r + r^2\theta_1 + \cdots + r^{n+1}\theta_m = \epsilon$, where r is the polar distance, the θ_i polynomials in $\sin \omega$, $\cos \omega$, ϵ a small positive constant, and where along γ the difference $\rho'(\omega) - r'(\omega)$ has a fixed

sign. Thus γ is a curve without contact in the sense of Poincaré. If a suitable curve γ exists the origin is a focus; otherwise it is a center. Furthermore the number of conditions necessary to determine which of the two occurs is finite. The author's results include as special cases those of Poincaré [Oeuvres, v. 1, p. 95] and Liapounoff [Problème Générale de la Stabilité du Mouvement, reprinted as Ann. of Math. Studies, no. 17, Princeton University Press, 1947; cf. these Rev. 9, 34]. *S. Lefschetz* (Princeton, N. J.).

Haag, Jules. Sur l'existence et la stabilité des solutions périodiques de certains systèmes différentiels. Ann. Sci. École Norm. Sup. (3) 65, 299–335 (1948).

A study of the existence and stability of periodic solutions of a system of differential equations $\dot{x}_i = f_i(x, t)$. The author begins with a study of linear equations, and extends his results to nonlinear equations by the classical perturbation method. The conjecture on page 315 that the absolute value of the characteristic roots of the variational equation must be less than one for stability is not true.

F. Bohnenblust (Pasadena, Calif.).

Sansone, Giovanni. Sopra una classe di equazioni di Liénard prive di integrali periodici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 156–160 (1949).

The author considers the Liénard equation $\ddot{x} + f(x)\dot{x} + x = 0$, where $\dot{x} = dx/dt$. A more general result than the author's can be proved as follows. Let $F(x) = \int_0^x f(x)dx$. Following Liénard let $\dot{x} = y - F(x)$. Then $\dot{y} = -x$ and the origin is the only singular point in the (x, y)-plane. Let $xF(x) < 0$ for $x \neq 0$ and let $F(x) = o(|x|)$ for large $|x|$. If $x = r\cos\theta$ and $y = r\sin\theta$ then $r\dot{r} = -xF(x) > 0$ for $x \neq 0$ and

$$\dot{\theta} = -1 + \sin\theta \cdot F(r\cos\theta)/r.$$

Thus r is monotone increasing (since $x \equiv 0$ over an interval of t implies $x \equiv 0$ for all t). Since no limit cycles can exist with r monotone increasing we see that $r(t) \to \infty$ as $t \to +\infty$ and $r(t) \to 0$ as $t \to -\infty$. For large r we have $\dot{\theta} = -1 + o(1)$ and thus x and y are oscillatory as $t \to +\infty$. [The author assumes $f(x) < 0$ for $a < x < b$, where $a < 0$ and $b > 0$, he assumes $f(x) > 0$ for $x > b$ and $x < -a$ and he further assumes $xF(x) < 0$ for $x \neq 0$ (which now implies that $F(x) = O(1)$ as $|x| \to \infty$). His result is that $x(t)$ and $\dot{x}(t) \to 0$ as $t \to -\infty$ and that $\limsup x(t) = +\infty$ and $\liminf x(t) = -\infty$ as $t \to +\infty$.]

N. Levinson (Cambridge, Mass.).

Minorsky, N. Energy fluctuations in a van der Pol oscillator. J. Franklin Inst. 248, 205–223 (1949).

The author considers the periodic solution of the van der Pol equation (V) $\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$ in the cases: (I) $\epsilon \ll 1$, (II) $\epsilon \gg 1$. Introducing polar coordinates (r, θ) and "energy" $\rho = r^2$, (V) is transformed into an equivalent pair of first order equations, $\dot{\rho} = f(\rho, \theta, \epsilon)$, $\dot{\theta} = g(\rho, \theta, \epsilon)$. In case (I) the method of perturbations is applied to obtain the initial terms of ϵ-power-series for θ and ρ as functions of time t, and for ρ as a function of θ; the arbitrary constants are evaluated by eliminating secular terms. The results are compared with the linear oscillator case, $\epsilon = 0$. In case (II) we have the "relaxation" phenomenon, a rapid change in direction of the phase trajectory near $\theta = 0$. The author obtains an expansion in powers of θ which enables him to discuss the change in energy as θ passes through the value zero. *J. G. Wendel* (New Haven, Conn.).

Reuter, G. E. H. Subharmonics in a nonlinear system with unsymmetrical restoring force. Quart. J. Mech. Appl. Math. 2, 198–207 (1949).

Poincaré's method of small parameters is applied to $\ddot{y} + \lambda\epsilon^2\dot{y} + (1 + \mu\epsilon^2)y(1 + \epsilon y) = 3\nu\epsilon\cos 2t$ for small ϵ and for λ, μ, ν constant. *N. Levinson* (Cambridge, Mass.).

Saltykow, Nicolas. Problèmes d'intégration d'équations aux différentielles totales. C. R. Acad. Sci. Paris 228, 1913–1915 (1949).

A la suite de la solution par l'auteur du problème d'intégration des équations aux différentielles totales de la forme $dx_{m+\nu} = a'_{\nu i}x_{m+i}dx_k$ ($k = 1, 2, \cdots, m$; $i = 1, 2, \cdots, n - m$; $\nu = 1, 2, \cdots, n - m$) où les a sont des constantes [mêmes C. R. 225, 520–521 (1947); ces Rev. 9, 186] Vessiot avait remarqué que cette solution avait besoin d'être complétée sur un point, et avait indiqué une autre méthode évitant cette objection [mêmes C. R. 226, 289–291 (1948); ces Rev. 9, 354]. L'auteur compare les avantages des différent procédés, et insiste en particulière sur le moyen de compléter par différentiation le nombre des intégrales distinctes, si les caractéristiques ne permettent point de les obtenir en nombre suffisant. *M. Janet* (Paris).

✱**Bilharz, Herbert.** Partielle Differentialgleichungen erster Ordnung und Pfaffsches Problem. Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 1–9, Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

A review of work by F. Engel [J. Reine Angew. Math. 180, 73–85 (1939)], E. Schmidt [Monatsh. Math. Phys. 48, 426–432 (1939); these Rev. 1, 76], Wei-Liang Chow [Math. Ann. 117, 98–105 (1939); these Rev. 1, 313], O. Perron [Math. Ann. 117, 687–693 (1941); Math. Z. 48, 136–172 (1942); these Rev. 3, 122; 5, 30], and E. Kamke [Math. Z. 49, 256–284 (1943); these Rev. 5, 183; Differentialgleichungen reeller Funktionen, 2d ed., Leipzig, 1944; Differentialgleichungen, Lösungsmethoden und Lösungen, v. 1, 3d ed., Akademische Verlagsgesellschaft, Leipzig, 1944; these Rev. 9, 33]. *J. M. Thomas.*

Bouligand, Georges. Sur les principes géométriques de la théorie des équations aux dérivées partielles. Ann. Soc. Polon. Math. 20 (1947), 229–240 (1948).

The purpose of this paper seems to be to reconsider the theory of partial differential equations of first order: $f(x, y, z, p, q) = 0$, starting from geometrical principles. A contingent integral of $f = 0$ is defined as a surface S such that at every point of S rays of the contingent of S do not lie in the inner part of the cone Γ but at least one of them lies on the surface of Γ, where Γ is the cone (assumed to be convex) of the elements of contact p, q satisfying $f = 0$ at the point x, y, z. The author quotes two results relating to contingent integrals proved previously by himself [Mém. Soc. Roy. Sci. Liège (3) 19, no. 10 (1934)] and A. Marchaud [Compositio Math. 3, 89–127 (1936)]. On the other hand, a surface is called a paratangent integral when its paratangent at every point is reduced to a plane which must be tangent to the cone Γ of $f = 0$. The essential part of this paper consists of showing the relation between the integrals of $f = 0$ in sense of Cauchy and paratangent integrals of the linear equation

$$(\lambda) \quad A_z\frac{\partial u}{\partial y} - B_z\frac{\partial u}{\partial x} + (BA_z - AB_z)\frac{\partial u}{\partial z}$$
$$+ (A_y - B_z + A_zB - B_zA) = 0$$

in the space of four dimensions $Oxyzu$, when we replace $f=0$ by the system $(S): p=A(x, y, z, u)$, $q=B(x, y, z, u)$ which represents the curve defined by $f=0$ in the plane (p, q) depending on three parameters x, y, z. This relation is as follows. There exists a one-parameter family of surfaces σ on a hypersurface $u=u(x, y, z)$ which is a paratingent integral of (λ), such that the projections of the surfaces σ on the space $u=0$ are integrals of $f=0$. This relation holds under some conditions for A and B. It is then remarked that these considerations can be extended to a system of equations $f_1=f_2=\cdots=f_n=0$ with n unknown functions $z_1, z_2, \cdots, z_n$ of two variables x, y, and that this general notion is useful for the recent researches of G. Llensa on the properties of derivability related to triply orthogonal systems [C. R. Acad. Sci. Paris 220, 297–298 (1945); 222, 845–847 (1946); these Rev. 7, 173, 481].

A. Kawaguchi (Sapporo).

∗Pinl, Max. **Partielle Differentialgleichungen zweiter und höherer Ordnung.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 21–45. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This bibliography of work done in Germany on the theory of partial differential equations will prove useful to those who were unable to obtain the relevant periodicals at the time when they appeared.   E. T. Copson (Dundee).

∗Buchholz, Herbert. **Spezielle Randwertaufgaben.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 47–52. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Berezin, I. S. **On Cauchy's problem for linear equations of the second order with initial conditions on a parabolic line.** Mat. Sbornik N.S. 24(66), 301–320 (1949). (Russian)

Let G be an open set in the half plane $y>0$, whose closure $\bar{G}$ contains an interval $a \leq x \leq b$ of the x-axis. The author considers the equation

$$(*) \quad z_{yy}=y^{\alpha}k^2(x,y)z_{xx}+a(x,y)z_x+b(x,y)z_y+c(x,y)z+f(x,y),$$

where k never vanishes on $\bar{G}$ and is twice continuously differentiable, the real coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$, $f(x, y)$ are once continuously differentiable on $\bar{G}$, and the constant $\alpha>0$. The Cauchy data (given on the parabolic line $y=0$) are $z(x, 0)=\varphi(x)$, $z_y(x, 0)=\psi(x)$ $(a \leq x \leq b)$, where φ and ψ have three continuous derivatives. A uniqueness and existence theorem is given in section 1, under the additional assumption that $\alpha<2$. The proof is carried out by means of an equivalent system of three integral equations. In section 2, again under the assumption that $0<\alpha<2$, it is shown that the Cauchy problem considered is correctly posed, i.e., that given $\epsilon>0$ there exists $\eta(\epsilon)>0$ such that if $\bar{\varphi}$ and $\bar{\psi}$ are thrice continuously differentiable on $a \leq x \leq b$, and

$$|\varphi(x)-\bar{\varphi}(x)|, \quad |\varphi'(x)-\bar{\varphi}'(x)|, \quad |\varphi''(x)-\bar{\varphi}''(x)| < \eta(\epsilon),$$
$$|\psi(x)-\bar{\psi}(x)|, \quad |\psi'(x)-\bar{\psi}'(x)|, \quad |\psi''(x)-\bar{\psi}''(x)| < \eta(\epsilon),$$

then $\bar{z}$, the solution of $(*)$ satisfying the Cauchy data $\bar{z}(x, 0)=\bar{\varphi}(x)$, $\bar{z}_y(x, 0)=\bar{\psi}(x)$, also satisfies

$$|z(x, y)-\bar{z}(x, y)| < \epsilon.$$

Section 3 contains an example of an equation of type $(*)$, with $\alpha>2$, for which Cauchy's problem is not correctly posed.   J. B. Diaz (Providence, R. I.).

Pailloux, Henri. **Choc longitudinal d'une barre prismatique.** C. R. Acad. Sci. Paris 228, 2006–2008 (1949).

A series solution of the one-dimensional wave equation, $a^2 u_{xx}=u_{tt}$, is given for boundary conditions $u(0, t)=0$, $u_x(1, t)=-c^2 u_{tt}(1, t)$ and initial conditions $u(x, 0)=0$, $0 \leq x \leq 1$; $u_t(x, 0)=0$, $0 \leq x<1$; $u_t(x, 0)=v$, $x=1$, where a, c and v are constants. An interesting feature of the paper is that the series may be summed.   R. D. Mindlin.

Smirnov, M. M. **Functional-invariant solutions of the wave equation.** Doklady Akad. Nauk SSSR (N.S.) 67, 977–980 (1949). (Russian)

A function u is said to be a functional-invariant solution of an equation

$$\sum_{i,k=1}^{n} a_{ik}(x_1,\cdots,x_n)u_{x_i x_k}+\sum_{i=1}^{n}b_i(x_1,\cdots,x_n)u_{x_i}+c(x_1,\cdots,x_n)u=0,$$

provided $F(u)$ is a solution of the equation for arbitrary F. The functional-invariant solutions of the wave equation $u_{xx}+u_{yy}=a^{-2}u_{tt}$ are given by the formula [V. Smirnov and S. Sobolev, Acad. Sci. URSS. Publ. [Trudy] Inst. Séismolog, no. 20 (1932)]

$$x+f(u(x, y))y+a\{1+f^2(u(x, y))\}^{1/2}t+h(u(x, y))=0,$$

where f and h are arbitrary functions. S. Sobolev [Trudy Fiz.-Mat. Inst. Steklov., Otd. Mat., no. 5, 259–264 (1934)] showed that these are all the functional-invariant solutions for this wave equation. N. P. Erugin is said to have shown that for the wave equation $u_{xx}+u_{yy}+u_{zz}=a^{-2}u_{tt}$ the corresponding Smirnov-Sobolev formula

$$x+f(u)y+g(u)z+a\{1+f^2(u)+g^2(u)\}^{1/2}t+h(u)=0$$

does not exhaust all invariant solutions (although it gives all real-valued ones) and to have exhibited two new classes of complex-valued functional-invariant solutions. The author consider the wave equation

$$(*) \quad u_{x_1 x_1}+u_{x_2 x_2}+u_{x_3 x_3}+u_{x_4 x_4}=a^{-2}u_{tt},$$

and finds that the Smirnov-Sobolev formula

$$x_1+f(u)x_2+g(u)x_3+e(u)x_4$$
$$+a\{1+f^2(u)+g^2(u)+e^2(u)\}^{1/2}t+h(u)=0,$$

which gives all real-valued functional-invariant solutions and part of the complex ones, has to be supplemented by nine new classes of complex-valued functional-invariant solutions. The problem of determining all functional-invariant solutions of $(*)$ is readily seen to be equivalent to finding all solutions common to $(*)$ and to the first order partial differential equation

$$u_{x_1}^2+u_{x_2}^2+u_{x_3}^2+u_{x_4}^2=a^{-2}u_t^2.$$

The procedure consists in obtaining a complete integral of this first order equation and then sifting out of it all solutions of $(*)$.   J. B. Diaz (Providence, R. I.).

Difference Equations, Special Functional Equations

Krull, Wolfgang. **Bemerkungen zur Differenzengleichung $g(x+1)-g(x)=\varphi(x)$.** Math. Nachr. 1, 365–376 (1948).

For a given real function $\varphi(x)$ $(x>a)$ the general solution g of the difference equation $g(x+1)-g(x)=\varphi(x)$ is uninteresting, since it may take arbitrary values on the interval $a<x \leq a+1$. If, however, φ is the sum of a convex function φ_1, and a concave function φ_2, each satisfying the boundary

condition $\varphi_i(x+h) - \varphi_i(x) \to 0$ $(h>0; i=1, 2)$ as $x \to \infty$ (whence φ_1 is nonincreasing and φ_2 nondecreasing), and if, moreover, g is required to be convex or concave or to satisfy the boundary condition

$$h_1^{-1}[g(x+h_1) - g(x)] - h_2^{-1}[g(x) - g(x-h_2)] \to 0$$
$$(0 < h_1, h_2 \leq \epsilon)$$

as $x \to \infty$, then g is uniquely determined up to an additive constant (of indefinite integration) as follows:

$$g(x) = \int^x \varphi(u)du - \tfrac{1}{2}\varphi(x) + \Delta(x),$$

$$\Delta(x) = \sum_{k=0}^{\infty} \delta(x+k),$$

$$\delta(x) = \int_x^{x+1} \varphi(u)du - \tfrac{1}{2}[\varphi(x) + \varphi(x+1)],$$

where $\Delta(x) \to 0$ as $x \to \infty$ and satisfies [in correction of formula (6)] the inequality

$$\varphi_1(x+\tfrac{1}{2}) - \varphi_1(x) \leq 2\Delta(x) \leq \varphi_2(x+\tfrac{1}{2}) - \varphi_2(x).$$

Furthermore, if $a=0$ and $g(1)=0$, then g is the limit as $n \to \infty$ of the functions

$$g_n(x) = x\varphi(n) - \varphi(x) + \sum_{k=1}^{n}[\varphi(k) - \varphi(x+k)],$$

which may also be written [by way of correcting in the limit the last member of formula (14)]

$$g_n(x) = x\varphi(0) - \varphi(x) + \sum_{k=1}^{n}[(1+x)\varphi(k) - \varphi(x+k) - x\varphi(k-1)],$$

wherein the term $x\varphi(0)$ is to be formally cancelled. The differentiability and convexity of g is investigated. In the guiding example $\varphi(x) = \log x$ with solution $g(x) = \log \Gamma(x)$ the first of the above representations is the Stirling series and the second is the logarithm of the Euler product formula. *W. Gustin* (Bloomington, Ind.).

Ansoff, H. I. Stability of linear oscillating systems with constant time lag. J. Appl. Mech. 16, 158–164 (1949).

The equation $\theta''(t) + a\theta'(t) + b\theta(t) = c\theta^{(n)}(t-\gamma)$ with a, b, c and γ constants is treated heuristically. [The mathematical literature which treats the general linear difference-differential equation with constant coefficients is overlooked.]
N. Levinson (Cambridge, Mass.).

Leont'ev, A. F. Differential-difference equations. Mat. Sbornik N.S. 24(66), 347–374 (1949). (Russian)

The author investigates, in the complex plane, the form and general analytic properties of the solutions of equation (i) $M(f) \equiv \sum_{m=0}^{p}\sum_{n=1}^{t}a_{m,n}d^m f(x+h_n)/dx^m = 0$ $(0=h_1<h_2< \cdots <h_q)$, where the $a_{m,n}$ are constants. Let $L(x)$ be the characteristic function

$$L(x) = \sum_{m=0}^{p}\sum_{n=1}^{t}a_{m,n}x^m \exp(h_n x),$$

and let the set $\{\lambda_n\}$ of zeros of $L(x)$ be divided into the two classes $\{\lambda_n'\}: |\lambda_1'| \leq |\lambda_2'| \leq \cdots$; $\{\lambda_n''\}: |\lambda_1''| \leq |\lambda_2''| \leq \cdots$, where $\{\lambda_n'\}$ consists of those λ_n for which either $\Im(\lambda_n)>0$ or $\Im(\lambda_n)=0$, $\Re(\lambda_n)>0$. Further, let $y_n'(x), y_n''(x)$ be those solutions of (i), corresponding respectively to λ_n' and λ_n'', given by $y_n'(x) = x^j \exp(\lambda_n'x)$, $y_n''(x) = x^k \exp(\lambda_n''x)$, where λ_n', λ_n'' are of respective multiplicity $j+1$, $k+1$. Let D denote an infinite strip $-\infty \leq \alpha < \Im(x) < \beta \leq \infty$.

The following results are established. (1) If $f(x)$ is regular in a strip D and satisfies (i), then in D it has the representation $f(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\{\alpha_{n,i}y_i'(x) + \beta_{n,i}y_i''(x)\}$, the convergence being uniform on every closed bounded set in D. (2) There is the decomposition $f(x) = \varphi(x) + \psi(x)$, where the functions

$$\varphi(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\alpha_{n,i}y_i'(x), \quad \psi(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\beta_{n,i}y_i''(x)$$

are regular in the respective half-planes $\Im(x)>\alpha$, $\Im(x)<\beta$, and are solutions of (i). (3) The limits $\alpha_i = \lim_{n\to\infty}\alpha_{n,i}$ exist, $i=1, 2, \cdots$, and if series (*) $\sum_{i=1}^{\infty}\alpha_i y_i'(x)$ converges in some half-plane $\Im(x)>c$ its sum in this half-plane is $\varphi(x)$; correspondingly for $\psi(x)$. (4) If $\lambda_{n_k-1}' \neq \lambda_{n_k}'$, $\lambda_{n_k+p_k}' \neq \lambda_{n_k}'$, $\lambda_{n_k}' = \lambda_{n_k+r}'$ $(r=1, 2, \cdots, p_k-1)$, and $n=n_k+j$ $(j<p_k)$, and if $\delta = \lim\sup |\lambda_n'|^{-1}\log|\gamma_n|$, where

$$|\gamma_n| = \{(z-\lambda_n')^{p_k}/L(z)\}^{(p_k-j-1)}$$

evaluated at $z = \lambda_n'$ (the superscript p_k-j-1 indicating order of differentiation with respect to z), then series (*) converges in the half-plane $\Im(x)>\alpha+\delta$. (And functions $\varphi(x)$ exist for which the abscissa of convergence $\alpha+\delta$ cannot be lowered.) (5) For each ρ in $0 \leq \rho \leq \infty$ there are equations of form (i) for which $\delta = \rho$. (If the differences $h_2, \cdots, h_q$ are commensurable, then $\delta = 0$.) (6) There exists a sequence of partial sums obtained from series (*) by a permutation of its terms that converges to $\varphi(x)$ in every half-plane $\Im(x)>\alpha$ where $\varphi(x)$ is regular.

In the proofs equation (i) is actually replaced by (†) which is obtained by replacing h_n by ih_n in $M(f)$, so that the new characteristic function is $L(x)$, obtained by replacing h_n by ih_n in the original $L(x)$, and the strip D is replaced by the corresponding strip $D: -\infty \leq \alpha < \Re(x) < \beta \leq \infty$. Let $\bar{M}(f)$ be the differential-difference operator corresponding to the characteristic function $\bar{L}(x) = L(-x)$. The author studies the equation (**) $M_{0,\infty}(f) = \bar{M}\{M(f)\} = 0$, with characteristic function $L_{0,\infty}(x) = L(x)L(-x)$. Equation (**) is made the basis of his investigation because it has the useful properties $\lim_{\rho\to\infty}\rho^{-1}\log|L_{0,\infty}(\rho e^{i\varphi})| = h_q|\sin\varphi|$, $\varphi \neq 0, \pi$; $\lim\sup_{\rho\to\infty}\rho^{-1}\log|L_{0,\infty}(\pm\rho)| = 0$; $L_{0,\infty}(x) = x^{2r}\prod_{n=1}^{\infty}(1-x^2/\mu_n^2)$ $(r \geq 0)$, and at the same time all solutions of (†) are solutions of (**).

Let $L_{k,\infty}(x) = \prod_{n=k}^{\infty}(1-x^2/\mu_n^2)$ $(k=1, 2, \cdots)$, and let $\gamma_{k,\infty} = \int_0^{\infty e^{i\varphi}}L_{k,\infty}(t)e^{-zt}dt$ be its Borel associate. For $f(x)$ regular in a strip D define the operator

$$M_{k,\infty}(f) = (2\pi i)^{-1}\int_C \gamma_{k,\infty}(t-x)f(t)dt,$$

C being the boundary of the rectangle $|\Re(t-x)| \leq \epsilon$, $|\Im(t-x)| \leq h_q+\epsilon$, lying in D. For x on any bounded closed region in D it is shown that $\lim_{k\to\infty}M_{k,\infty}(f) = f(x)$ holds uniformly. Using this, if $z_i'(x), z_i''(x)$ are the solutions of (**) corresponding to the zeros $\mu_i, -\mu_i$ of $L_{0,\infty}(x)$, then a solution $f(x)$ of (**), regular in a strip D, has the representation

$$f(x) = P(x) + \lim_{n\to\infty}\sum_{i=1}^{n}\{\alpha_{n,i}z_i'(x) + \beta_{n,i}z_i''(x)\},$$

where P is a polynomial and where the convergence is uniform in every $D_1 \subset D$, D_1 being bounded and closed. Conversely, a function $\varphi(x)$, having such a representation (without P) uniform in every D_1, is a solution of (**). Many results (that cannot be summarized in short space) are obtained relative to equation (**), and these are shown to lead to results on equation (†), yielding the assertions (1) to (6) stated earlier. *I. M. Sheffer* (State College, Pa.).

Functional Analysis, Ergodic Theory

Wielandt, Helmut. Eigenwerttheorie. Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 85–98. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

A review of results in eigen-value theory appearing in German publications during the years 1939–1946. The following types are treated: eigen-value-problems (a) in abstract spaces, (b) for integrable functions, (c) for differentiable functions, (d) for a finite number of unknowns.

E. H. Rothe (Ann Arbor, Mich.).

Köthe, Gottfried. Funktionalanalysis, Integraltransformationen. Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 99–112. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

A brief survey of results in the field and period indicated.
M. M. Day (Urbana, Ill.).

Sheffer, I. M. On the theory of sum-equations. Bull. Amer. Math. Soc. **55**, 777–788 (1949).

This paper supplements the author's earlier paper [Trans. Amer. Math. Soc. **63**, 244–313 (1948); these Rev. **9**, 517] on sum-equations. By a sum-equation system is meant a system of the form (1) $\sum_{j=0}^{n} a_{p j} x_{p+j} = c_p$, $(p=0, 1, \cdots)$, with $a_{p0} \neq 0$. The system is called k-periodic if $a_{p+k,s} = a_{p,s}$ for all p and s, and is called homogeneous if $c_p = 0$. The type $((y_n))$ of a sequence $\{y_n\}$ is defined as $\limsup |y_n|^{1/n}$. Three kinds of systems are considered: (I) nonhomogeneous, k-periodic; (II) nonhomogeneous, not k-periodic; (III) homogeneous, k-periodic. For (I), assuming that for some $q>0$ $((c_n)) \leq q$ and (2) $((a_{pn})) < 1/q$ $(p=0, 1, \cdots)$, a solution $\{x_n\}$ with $((x_n)) = ((c_n))$ is found in the form (3) $x_n = \int_\gamma t^{n-1} R(t) dt$, γ being a suitable circle in the t-plane, and $R(t)$ being a certain rational combination of the functions $\sum_{i=0}^{\infty} c_i \omega^{-j_i t^{-i}}$, and (4) $\omega^{jp} \sum_{j=0}^{\infty} a_{pj} \omega^{j t^i}$ $(j, p=0, 1, \cdots, k-1; \omega = \exp(2\pi i/k))$. For (II) similar results are obtained under a special assumption designed to insure that the system (1) can be approximated effectively for large k by that k-periodic system which coincides with (1) for the first k equations. For (III) under assumption (2) a family F of solutions of type not exceeding q is exhibited in the form of a contour integral resembling (3); conclusions as to the linear independence of members of F are deduced from further assumptions that at some of the points t where the matrix of the functions (4) is singular that matrix is of rank $k-1$.   *W. Strodt.*

Matsuyama, Noboru. Linear topological spaces and its pseudo-norms. Tôhoku Math. J. (2) **1**, 14–21 (1949).

It is shown that the directed system D in the reviewer's definition of a pseudo-norm [Duke Math. J. **5**, 628–634 (1939); these Rev. **1**, 58] may be replaced either by a semi-join-lattice or a semi-meet-lattice.   *D. H. Hyers.*

Klee, V. L., Jr. Dense convex sets. Duke Math. J. **16**, 351–354 (1949).

The deficiency $\delta_L(E)$ of a linear subset E in a linear space L is the dimension of an arbitrary complement of E in L. (A) Let L be a linear space with a topology such that for fixed y, r the mapping $x \to y + rx$ is continuous in x. Suppose L has a dense linear subset of deficiency $N \geq 1$, and let $\aleph$ be a cardinal number satisfying either $\aleph = \exp(\aleph_0)$ or $\aleph \leq \max(\aleph_0, N)$. Then L can be expressed as a union of $\aleph$ pairwise disjoint convex dense subsets. This result is applied to L', an infinite-dimensional linear topological space $(y + rx$

continuous in all three variables at once) satisfying Hausdorff's first denumerability axiom, by means of: (B) A linear subset E of L' has a dense complement if and only if $\delta_{L'}(E)$ has at least the minimal cardinal number possible for a dense subset of L'.   *M. M. Day* (Urbana, Ill.).

Al'tman, M. Š. On biorthogonal systems. Doklady Akad. Nauk SSSR (N.S.) **67**, 413–416 (1949). (Russian)

The author discusses first some conditions on a sequence $\{y_i\} \subseteq B$, a Banach space, in terms of a biorthogonal sequence $\{F_i\} \subseteq B^*$ and a given basis in B so that $\{y_i\}$ shall be a basis in B. He applies the result to Hilbert space as follows. If $\{f_i\}$, $\{g_i\}$ are biorthogonal and $\{f_i\}$ a basis in H, $\{f_i\}$ is called a Bessel basis if $\sum (x, g_i)^2 < \infty$ for all $x \epsilon H$; $\{f_i\}$ is called a Riesz basis if in addition for each sequence $\{a_i\}$ such that $\sum |a_i|^2 < \infty$ there is an x in H with $(x, g_i) = a_i$ for all i. It is stated that in order that $\{f_i\}$ biorthogonal to $\{g_i\}$ be a Bessel basis it is necessary and sufficient that there exist a constant M such that for all $x \epsilon H$

$$\sup_n \left\| \sum_{i \leq n} (x, g_i)(\varphi_i - f_i) \right\| \leq M \|x\|,$$

where $\{\varphi_i\}$ is an arbitrary Riesz basis assigned in advance.
M. M. Day (Urbana, Ill.).

Nachbin, Leopoldo. On the Han-Banach theorem. Anais Acad. Brasil. Ci. **21**, 151–154 (1949).

Abstracting the property of the real numbers on which rests the usual proof of the extensibility of linear functionals, the author singles out the class BI of normed linear spaces in which any family of balls of which any two members meet has a nonvoid intersection. Besides generalizing the aforementioned extension theorem to operators with values in a space (BI), the author states, conversely, that if the extensions can be carried out in all cases, then the range must be a space (BI). Spaces (BI) which have an extreme point of the unit ball are characterized as the spaces $C(X, R)$ of continuous real-valued functions on an extremally disconnected compact Hausdorff space X. Details of proofs are omitted.   *R. Arens* (Los Angeles, Calif.).

Yood, Bertram. Additive groups and linear manifolds of transformations between Banach spaces. Amer. J. Math. **71**, 663–677 (1949).

Soient X, Y deux espaces de Banach, $A(X, Y)$ l'espace des applications linéaires continues de X dans Y. Étant donné un sous-groupe additif G de $A(X, Y)$, l'auteur considère le sous-espace $\mathfrak{M}(G)$ de X où s'annulent tous les $u \epsilon G$, et le sous-espace $\mathfrak{N}(G)$ de Y engendré par les $u(X)$ pour $u \epsilon G$; munissant $A(X, Y)$ de la topologie "finie" (topologie de la convergence simple, quand on considère X et Y comme des espaces discrets), il cherche des conditions moyennant lesquelles l'adhérence de G dans $A(X, Y)$ est l'ensemble G' des v tels que $v^{-1}(0) \supset \mathfrak{M}(G)$ et $v(X) \subset \mathfrak{N}(G)$. Pour cela il associe à G deux anneaux d'endomorphismes; l'un, $R_l(G)$, formé des endomorphismes continus w de $\mathfrak{N}(G)$ tels que $wu \epsilon G$ pour tout $u \epsilon G$; l'autre, $R_r(G)$, formé des endomorphismes continus v de $X/\mathfrak{M}(G)$ tels que $uv \epsilon G$ pour tout $u \epsilon G$; une condition suffisante pour que G' soit l'adhérence de G est alors que $R_l(G)$ soit dense dans l'espace $A(\mathfrak{N}(G), \mathfrak{N}(G))$; condition analogue portant sur $R_r(G)$. Il prouve à cette occasion que si B est une sous-algèbre une fois transitive de $A(X, X)$, B est dense. [En réalité la propriété est vraie en supposant seulement que B est un anneau une fois transitif: en effet, le commutant C de B dans l'anneau de tous les endomorphismes du groupe additif X est un corps qui contient les scalaires; si $x \neq 0$ dans X,

les $u(x)$, où $u \epsilon C$, forment un sous-espace vectoriel F de X; soit B' l'ensemble des restrictions à F des $v \epsilon B$ tels que $v(x) \epsilon F$; comme B est une fois transitif, et F de dimension 1 sur C, B' est l'ensemble de tous les endomorphismes de F considéré comme espace vectoriel sur C, donc B' est un corps isomorphe à l'opposé de C; comme on peut munir B' d'une structure d'algèbre normée (B' étant formée d'endomorphismes continus de l'espace normé F), B' est identique au corps des nombres complexes d'après le théorème de Gelfand-Mazur, donc il en est de même de C.]

Les résultats de l'auteur lui permettent, sous les mêmes conditions, d'obtenir les adhérences de G pour les topologies fortes et faibles dans $A(X, Y)$. Il étudie en particulier les groupes G pour lesquels $wu \epsilon G$ pour tout $w \epsilon A(Y, Y)$ et tout $u \epsilon G$ (respectivement $uv \epsilon G$ pour $v \epsilon A(X, X)$ et $u \epsilon G$), qui généralisent la notion d'idéal. Enfin, il prouve que le commutant d'un idéal dans $A(X, X)$ se réduit au corps des scalaires.   *J. Dieudonné* (Nancy).

Adel'son-Vel'skiĭ, G. M. Spectral analysis of a ring of bounded linear operators on Hilbert space. Doklady Akad. Nauk SSSR (N.S.) **67**, 957–959 (1949). (Russian)

The following construction is announced. Suppose that R is a self-adjoint algebra of bounded linear operators in a Hilbert space $\mathfrak{H}$ and ξ an element of $\mathfrak{H}$ such that $R\xi$ is dense in $\mathfrak{H}$. Let C be a maximal commutative self-adjoint subalgebra of the commuting algebra R' of R. According to Gelfand and Neumark [Rec. Math. [Mat. Sbornik] N.S. 12(54), 197–213 (1943); these Rev. **5**, 147] the elements Ψ of C can be considered as the continuous functions $\Psi(M)$ on the maximal ideals M of C. Therefore the inner product $(\Psi\xi, \xi)$ equals $\int \Psi(M)d\sigma(M)$ with a suitable measure σ. Application of the Radon-Nikodym theorem shows that for $X \epsilon R$ one has $(X\Psi\xi, \xi) = \int \Psi(M)D(X, M)d\sigma(M)$. For fixed M, $D(X, M)$ is a linear function on R and determines a representation of R by operators in some space $\mathfrak{H}_M$. The author indicates next how to associate with any $\eta \epsilon \mathfrak{H}$ a vector-valued function η_M (with values in $\mathfrak{H}_M$) such that $(\eta, \eta') = \int (\eta_M, \eta'_M)d\sigma(M)$ and announces a characterization of those functions η_M which correspond to elements η of $\mathfrak{H}$. This is closely related to von Neumann's work on generalized direct sums of Hilbert spaces [Ann. of Math. (2) **50**, 401–485 (1949); these Rev. **10**, 548] which was probably not available to the author. The paper closes with the following statement: "The problem of the irreducibility of the representations of the algebra R by algebras of bounded linear operators on the spaces $\mathfrak{H}_M$ requires further investigation, the results of which are to be published in the future."

F. I. Mautner (Cambridge, Mass.).

Tseng, Ya. Yu. Generalized inverses of unbounded operators between two unitary spaces. Doklady Akad. Nauk SSSR (N.S.) **67**, 431–434 (1949). (Russian)

A theory of generalized inverses (g.i.) of closed operators from a unitary space to another unitary space is developed; this generalizes the Moore-Barnard theory for modular matrices and other theories. If A is a linear operator, R is a g.i. of A if the range of each of A and R is contained in the domain of the other, and if the product of the two operators is the projection operator of the closure of the domain of the first operator in the product. The following existence criterion is given: if A is an operator with dense domain, it has a g.i. if and only if the orthogonal projection of its domain on the closure of its nulmanifold lies in the nulmanifold. There then exists a unique maximal g.i. R of A, which has

the property that for any g in the range of A or its orthogonal complement $\|Ax - g\|$ attains its minimum value for $x = Rg$ and Rg is the smallest x, in norm, for which this minimum is attained. Further relations of operators and their g.i. are stated.   *J. L. B. Cooper* (London).

Tseng, Ya. Yu. Properties and classification of generalized inverses of closed operators. Doklady Akad. Nauk SSSR (N.S.) **67**, 607–610 (1949). (Russian)

A number of theorems on generalised inverses of closed operators are stated, in continuation of the article reviewed above. A closed operator A has a unique closed g.i., R, and A^* and R^* are g.i. Necessary and sufficient conditions for existence of a bounded g.i. are stated. Closed operators are divided into four classes, depending on whether domain and range are closed or not, and each of these into four subclasses, depending on whether the ranges of the operator and its adjoint are dense in the spaces in which they exist or not. Properties of the generalised inverse and of the operator are stated for each case.   *J. L. B. Cooper*.

Kaplansky, Irving. Normed algebras. Duke Math. J. 16, 399–418 (1949).

Some new results about normed algebras are proved. Let A be a Banach algebra without radical; if every square in A has a quasi-inverse, A is commutative. Let A be a Banach algebra in which the operators of the "regular representation" are completely continuous; then the structure space of A (in the sense of Jacobson) is discrete. The author gives a simple proof of the following theorem, asserted without proof by Gelfand and Neumark [Rec. Math. [Mat. Sbornik] N.S. 12(54), 197–213 (1943); these Rev. **5**, 147]: let A be a uniformly closed self-adjoint algebra of operators in a Hilbert space; then every closed two-sided ideal I in A is self-adjoint, and A/I can be isometrically imbedded into the ring of operators of a Hilbert space. Let A be the normed algebra of completely continuous operators in a Hilbert space H; then every closed right ideal in A is the annihilator of some closed subspace of H. [Note that it follows from the preceding result that A has only one irreducible unitary representation, namely the trivial one.] The author studies "central" algebras, i.e., algebras in which primitive ideals correspond in one-to-one fashion to their intersections with the center of the algebra. The main result is as follows: let A be a Banach algebra with an involution which satisfies $\|x^*\| = \|x\|$, $\|x^*x\| = \|x\|^2$; if A is central, then the structure space of A is locally compact, and homeomorphic with the structure space of the center of A. [Similar facts arise when A is a ring of operators of "finite class" in the sense of Dixmier, C. R. Acad. Sci. Paris **228**, 152–154 (1949); these Rev. **10**, 381.]   *R. Godement* (Nancy).

Bourgin, D. G. Approximately isometric and multiplicative transformations on continuous function rings. Duke Math. J. 16, 385–397 (1949).

Let $C(X_i)$ be the normed ring of all bounded real-valued continuous functions on the space X_i, $i = 1, 2$. In an earlier paper Hyers and Ulam [Ann. of Math. (2) **48**, 285–289 (1947); these Rev. **8**, 588] showed that if the X_i are compact metric and if there exists a homeomorphism T of $C(X_1)$ onto $C(X_2)$ which is an ϵ-isometry, then there exists a true isometry U of $C(X_1)$ onto $C(X_2)$ such that $\|U(f) - T(f)\| \leq 21\epsilon$ for all f in $C(X_1)$. Also the existence of T implies that X_1 is homeomorphic to X_2. The author extends this result to the

case where the X_i are completely regular and to a generalized ϵ-isometry T which need not be single-valued or a homeomorphism. Also the existence of T implies the homeomorphism of the X_i if they are compact spaces.

The author also investigates "approximately multiplicative" transformations defined from $C(X_1)$ onto $C(X_2)$. Under suitable hypotheses the existence of such a one-to-one transformation implies that X_1 is homeomorphic to X_2. These hypotheses may be satisfied by transformations not multiplicative. This leads the author to consider conditions under which "approximately multiplicative" transformations are multiplicative. For detailed statements of these results reference must be made to the paper itself.

B. Yood (Ithaca, N. Y.).

Levitan, B. M. The application of generalized displacement operators to linear differential equations of the second order. Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 3–112 (1949). (Russian)

Topics covered in this exposition are the author's axiomatic characterization of generalised displacement operators (g.d.o.'s), the spectral analysis of such operators, and corresponding generalisations of positive definite and almost periodic sequences and functions. Special attention is given to cases in which the solutions of $y'' - (\rho(x) - \lambda)y = 0$ with $y'(0) - hy(0) = 0$ play the role of trigonometric functions in the ordinary theory. This initial condition is indicated as being more general than that in the author's previous work. Chapter I. Consider numerical functions $f(t)$ defined over a locally compact topological group with second axiom of countability, and positive completely additive measure $m(t)$, $f(t)$ belonging to L_p over Z. Definition: T_t^s is a g.d.o. if (I) there is a unit e of Z such that $T_t^e f(t) = f(t)$, $T_e^s f(t) = f(s)$, (II) T_t^s is a linear operator, (III) "associativity," i.e., $T_\tau^s T_t^s f(t) = T_t^\tau T_\tau^s f(t)$, (IV) T_t^s and its adjoint $\tilde{T}_t^s$ are suitably bounded on L_p, (V) T_t^s and $\tilde{T}_t^s$ depend continuously on s. In special cases the T_t^s may be commutative and normal as well [cf. the author's different formulation in Rec. Math. [Mat. Sbornik] N.S. 16(58), 259–280 (1945); these Rev. 7, 254]. Examples: (i) Z the set of integers, L_p the set of sequences $\{x_k\}_{-\infty}^{\infty}$ with $\sum |x_k|^p < \infty$, the g.d.o.'s being given by $T_s^p x_q = \sum_r c_{pq}^r x_r$, the c_{pq}^r being a cubic matrix [Doklady Akad. Nauk SSSR (N.S.) **58**, 977–980 (1947); these Rev. 9, 347]; (ii) write $T_x^y f$ for the function $F(x, y)$ such that $F_{xx} - \rho(x)F = F_{yy} - \rho(y)F$, $F(x, 0) = f(x)$, $F_y(x, 0) - hF(x, 0) = 0$; then T_x^y is shown, under conditions, to be a g.d.o.

Chapter II. Resolutions of the identity are established for integral operators A of the form $A\varphi = \int \tilde{T}_s^t f(t) \varphi(s) dm(s)$, where T^s is a family of g.d.o.'s regarded as operators on the Hilbert space of functions of L_2 over Z. The spectral resolution is established by Carleman's method of first taking the integral over a compact subset X of Z. The case of a commutative family of such operators A is covered by the theory of von Neumann [Math. Ann. 102, 370–427 (1929); cf. also Levitan, Rec. Math. [Mat. Sbornik] 19(61), 407–427 (1946); these Rev. 9, 7]. It is shown that if the family of g.d.o.'s T^s is commutative, then so are the associated integral operators A, and hence are deduced for the commutative family T^s certain generalisations of the Parseval formulae.

Chapter III. It is shown that this theory for commutative g.d.o.'s, including the Parseval formulae, applies to certain cubic matrices and to the Sturm-Liouville case, the original investigations being due to A. Haar [Math. Z. 31, 769–798

(1930)] and H. Weyl [Math. Ann. 68, 220–269 (1910)]. [See also B. M. Levitan, Rec. Math. [Mat. Sbornik] N.S. 17(59), 9–44, 163–192 (1945); these Rev. 7, 254; 8, 157.]

Chapters IV–VII. Generalised positive definite and almost periodic sequences and functions. These chapters are mostly occupied with proving results previously announced by the author without proof [Doklady Akad. Nauk SSSR (N.S.) **58**, 977–980, 1593–1596 (1947); these Rev. 9, 347]. Definitions are given of positive definite and almost periodic functions and sequences with respect to a g.d.o., but the detailed discussion is confined to the special cases treated in these papers. However for the case of the Sturm-Liouville operator there is the above-mentioned generalisation of the initial condition.

F. V. Atkinson (Ibadan).

Cameron, R. H., and Martin, W. T. The transformation of Wiener integrals by nonlinear transformations. Trans. Amer. Math. Soc. 66, 253–283 (1949).

The authors study the behavior of Wiener integrals under transformations of the form $T: y(t) = x(t) + \Lambda(x|t)$, where $\Lambda(x|t)$ is a functional depending on the function x and the number t, and satisfying certain smoothness conditions. Here x ranges over a measurable subset of the space C of continuous functions on $0 \leq t \leq 1$ which vanish when $t = 0$. The smoothness conditions assumed on $\Lambda(x|t)$ require it to have a Volterra derivative $K(x|t, s)$ such that

$$\delta\Lambda = \frac{\partial}{\partial h}\Lambda(x + h\delta x|t)]_{h=0} = \int_0^1 K(x|t, s)\delta x(s)ds.$$

The Fredholm determinant

$$D(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \cdots \int_0^1 (n) \begin{vmatrix} K(x|t_1, t_1) \cdots K(x|t_n, t_1) \\ \cdots \\ K(x|t_1, t_n) \cdots K(x|t_n, t_n) \end{vmatrix} dt_1 \cdots dt_n$$

of K is thought of as analogous to the functional determinant or Jacobian of an n-dimensional transformation in the development of the theory. The main result of the paper is the establishment, under appropriate conditions, of the equation

$$\int_T^W F(y)d_w y = \int_\Gamma^W F\{x + \Lambda(x|\cdot)\} \exp\left\{-2\int_0^1 \left[\frac{\partial}{\partial t}\Lambda(x|t)\right]dx(t) - \int_0^1 \left[\frac{\partial}{\partial t}\Lambda(x|t)\right]^2 dt\right\} |D(x)|d_w x.$$

First certain smoothness conditions on Λ are introduced which imply the existence of its Volterra derivative K, and their properties are discussed. Then the one-to-one-ness of T is treated, along with its approximating transformations. Then an approximating "n-dimensional" form of

$$(1) \quad \int_{TF}^C F(y)d_w y = \int_\Gamma^C F(Tx)|D(x)| \times \exp\left\{-\int_0^1 \frac{d}{dt}[Tx(t) - x(t)]d[Tx(t) + x(t)]\right\}d_w x$$

is established and the passage to the limit on n is justified. The restriction on K,

$$\left\{\int_0^1 [K(x|t, s)]^2 ds\right\}^{\frac{1}{2}} \leq \lambda < 1,$$

assumed at first, is removed in two local theorems. The final theorem establishes (1) in the large, and this is wrought by the application of a local theorem in a countable set of neighborhoods and then the use of the complete additivity of the Wiener integral.

C. Hatfield, Jr.

Fantappiè, Luigi. **I funzionali derivati del determinante e del nucleo risolvente di un nucleo dato.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 329–333 (1948).

Es werden die Ableitungen, und anschliessend die Variationen, der ganz transzendenten Funktion $D(\lambda)$, des Lösungskernes $R(\lambda; z, t)$ und des sogenannten "assozierten" Kernes $\Gamma(\lambda; z, t) = 1/(1-zt) + \lambda R(\lambda; z, t)$ berechnet. Diese Funktionen, welche sich in der Theorie der Integralgleichungen zweiter Art natürlich ergeben, erweisen sich im analytischen Gebiet als nicht lineare, analytische Funktionale des Kernes $K(z, t)$ der gegebenen Gleichung. Es wird hervorgehoben, dass ihre abgeleiteten Funktionale und ihre Variationen mit Hilfe der Funktionale selbst in endlicher Form ausdrückbar sind. *H. G. Haefeli* (Boston, Mass.).

Pellegrino, Franco. **Su alcune proprietà fondamentali delle regioni funzionali non lineari.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 333–339 (1948).

Die Arbeit untersucht Eigenschaften der offenen Gebiete R des Funktionenraumes der lokal-analytischen Funktionen, auf welchen ein nicht lineares und nicht lokal konstantes, analytisches Funktional definiert sein kann. Es wird gezeigt, dass jede abgeschlossene Punktmenge A_0 der komplexen Zahlkugel, mit welcher eine Umgebung $(A_0, \sigma_0) \subset R$ eines Punktes $y_0 \subset R$ (alle analytischen Funktionen y definiert über $A_0 \subset M_0$, sodass max $|y - y_0| < \sigma_0$ in A_0; M_0 Definitionsgebiet von y_0) gebildet werden kann, mit jeder andern abgeschlossenen Punktmenge A_1, die zu einer Umgebung $(A_1, \sigma_1) \subset R$ von y_0 oder eines beliebigen andern Punktes $y_1 \subset R$ gehört, Punkte gemeinsam haben muss. Es wird ferner gezeigt, dass der Durchschnitt aller A_k, die zu den Umgebungen eines festen Punktes gehören, nicht leer ist, und ein Beweis in Aussicht gestellt, dass dasselbe für den Durchschnitt überhaupt aller Mengen A_k gilt, mit denen Umgebungen in R gebildet werden können. Letzteres würde besagen, dass jedes solche Gebiet R in einem linearen Unterraum liegt, und eine natürliche Identifikation und starke Topologie zu definieren gestatten. Abgesehen davon ist es an sich interessant zu sehen, wie weit die schwer zu überblickende Forderung nach Analytizität die Existenzgebiete eines nicht linearen, analytischen Funktionals einschränken. *H. G. Haefeli* (Boston, Mass.).

Fomin, S. **On the theory of dynamical systems with continuous spectrum.** Doklady Akad. Nauk SSSR (N.S.) **67**, 435–437 (1949). (Russian)

Let X be the space of all sequences $\{x_n\}$, $n = 0, \pm 1, \pm 2, \cdots$, of real numbers and let T be the transformation defined by $T\{x_n\} = \{y_n\}$, where $y_n = x_{n+1}$. If a measure μ in X is invariant under T, the correlation function B of μ is defined, for every integer k, by $B(k) = \int_X x_0 x_k d\mu$. It is well known that a function B is the correlation function of some invariant measure μ if and only if it is positive definite, i.e., if and only if there is a measure m in the perimeter C of the unit circle such that $B(k) = \int_C \lambda^k dm(\lambda)$; the measure m is called the spectral function of μ. A measure μ is called normal if all its finite-dimensional sections are Gaussian. Theorem 1. If μ is a normal invariant measure in X, then a necessary and sufficient condition that the transformation T be weakly mixing is that the spectral function m of μ be nonatomic; hence, in particular, if m is nonatomic, then T is indecomposable. The spectral type (in the sense of Hellinger) of T is $M = \sum_{n=1}^{\infty} 2^{-n} m^{(n)}$ (where $m^{(1)} = m$ and $m^{(n+1)} = m^{(n)} * m$ for $n = 1, 2, \cdots$). Theorem 2. If M is any nonatomic measure

in C such that $M^{(n)}$ is absolutely continuous with respect to M, $n = 1, 2, \cdots$, then there exists a measure μ invariant under T and such that T is indecomposable and has spectral type M. *P. R. Halmos* (Chicago, Ill.).

Mil'man, D. **Multimetric spaces. Analysis of the invariant subsets of a multinormed bicompact space under a semigroup of nonincreasing operators.** Doklady Akad. Nauk SSSR (N.S.) **67**, 27–30 (1949). (Russian)

This note extends a sequence of notes on "functionally-defined" bicompact sets [same Doklady (N.S.) **57**, 119–122 (1947); **59**, 1045–1048, 1241–1244, 1397–1398 (1948); these Rev. **9**, 192, 449, 516, 449]. A linear space E is called multinormed if the topology in E is given by a set R of pseudo norms $\| \ \|_s$ [von Neumann, Trans. Amer. Math. Soc. **37**, 1–20 (1935), in particular, p. 18]. An operator U from E into E is called a contraction if, for every x in R and f, g in E, $\|Uf - Ug\|_s \leq \|f - g\|_s$. Assume that G is a semigroup of linear contractions of E and that K is a convex compact subset of E mapped onto itself by each U of G. Let A be the set of extreme points of K and let $K(G)$ be the set of fixed points of G in K. An invariant subset A_1 of A closed in A is called p-ergodic if its closed convex hull contains just one point of $K(G)$; a minimal p-ergodic set A_1 is called ergodic if its corresponding fixed point is an extreme point of $K(G)$, and the fixed point in an ergodic A_1 is also called ergodic. Under these hypotheses theorem 3 asserts: (a) the set of ergodic fixed points of G in K is dense in the set of extreme points of $K(G)$, so $K(G) \subseteq$ closed convex hull of the set of ergodic fixed points; (b) each minimal closed (in A) invariant subset of A is p-ergodic, so A can be decomposed into minimal p-ergodic parts. (That the first assertion of (b) implies the second is part of the first theorem of the present note.) *M. M. Day* (Urbana, Ill.).

Mil'man, D. P. **Extremal points and centers of convex bicompacta.** Uspehi Matem. Nauk (N.S.) **4**, no. 5 (33), 179–181 (1949). (Russian)

The material stated in this abstract consists mainly of generalizations of the results abstracted in the note reviewed above and of applications to an abstract form of dynamical systems. No proofs are given. *M. M. Day* (Urbana, Ill.).

Theory of Probability

Toranzos, Fausto I. **Elements of a definition of probability.** Revista Fac. Ci. Econ. Univ. Cuyo **1**, 6 pp. (1949). (Spanish)

Bassali, W. **Probability problems in nuclear chemistry. II.** Proc. Roy. Irish Acad. Sect. A. **52**, 191–201 (1949).

[The first part, by E. Schrödinger, appeared in the same Proc. Sect. A. **51**, 1–8 (1945); these Rev. **7**, 457.] It was shown in the first part that, under certain conditions, the probability density $p(r)$ of a fission's taking place at the point r satisfies the integral equation $\lambda p(r) = \int p(r') f(r, r') dv'$, where r and r' are position vectors to points of the body and $f(r, r')$ the probability density that a neutron produced at r scores a splitting hit at r'. The author shows that, to a certain approximation, $f(r, r') = A e^{-hs} s^{-2}$, where A and h are constants and $s = |r - r'|$. He applies the Rayleigh-Ritz method to the case "of very small neutron absorption" (essentially neglecting powers of k). Finally, he obtains con-

siderably better numerical results by the perturbation method and various refinements.   *W. Feller.*

Borel, Émile.   Le paradoxe de Saint-Pétersbourg.   C. R. Acad. Sci. Paris **229**, 404–405 (1949).

Borel, Émile.   Sur une propriété singulière de la limite d'une espérance mathématique.   C. R. Acad. Sci. Paris **229**, 429–431 (1949).

The classical Petersburg game is modified as follows. A true coin is tossed until it falls heads, but at most n times. Peter wins at heads. The stake at the kth trial is $(k+1)2^{k-1}$. It is shown that the game is "fair" and the passage to the limit $n \to \infty$ is investigated; trouble arises due to the circumstance that the expectation for the infinite game is of the form $0 \cdot \infty$ and hence the game is no longer necessarily "fair."   *W. Feller* (Ithaca, N. Y.).

Kawada, Yukiyosi.   On a characterization of multiple normal distributions.   Kōdai Math. Sem. Rep., no. 3, 1–2 (1949).

Apparently the author was unaware of a previous result by Kac [Amer. J. Math. **61**, 726–728 (1939); these Rev. **1**, 62].   *K. L. Chung* (Ithaca, N. Y.).

Pompilj, Giuseppe.   Sulla media geometrica e sopra un indice di mutabilità calcolati mediante un campione.   Mem. Soc. Ital. Sci. (3) **26**, 299–339 (1947).

L'auteur expose d'abord des généralites sur la méthode de la statistique mathématique et développe les exemples suivants : soit X un élément aléatoire pouvant prendre, avec probabilité $p_1, \cdots, p_k$ l'une ou l'autre des valeurs $a_1, \cdots, a_k$, et A_i la repetition, au cours de n épreuves indépendantes, de l'événement $X = a_i$; en supposant d'abord que les a_i sont des nombres réels positifs (x est donc une variable aléatoire ordinaire), l'auteur étudie la variable aléatoire $\mathfrak{M}_g = \prod a_i^{A_i/n}$, dont il évalue asymptotiquement l'espérance mathématique et l'écart moyen quadratique et dont il montre qu'elle est asymptotiquement laplacienne; en supposant que les a_i ne sont pas nécessairement des nombres (X est alors un élément aléatoire abstrait, que l'auteur appelle une "mutable aléatoire"), l'auteur calcule l'espérance mathématique et l'écart moyen quadratique de la variable aléatoire $\mathfrak{G} = \sum_i n^{-1} A_i (1 - A_i/n)$, dérivée de l'indice de mutabilité $g = \sum p_i(1 - p_i)$ de Gini [qu'on peut rapprocher de l'entropie $-\sum p_i \log p_i$].   *R. Fortet* (Caen).

Milicer-Gruzewska, Halina.   Sulla legge limite delle variabili casuali equivalenti.   Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) **2**, 25–33 (1948).

A sequence of random variables $\{X_n\}$ is here called equivalent if the joint distribution of any p-tuple $(X_{i_1}, \cdots, X_{i_p})$ (with $i_j \neq i_k$ if $j \neq k$) is the same. In this case

$$E\{X_{i_1}^{r_1} \cdots X_{i_p}^{r_p}\} = M(r_1, \cdots, r_p)$$

depends only on the integers r_j. It is assumed that $|M(r_1, \cdots, r_p)| < r! S^r$, where $r = r_1 + \cdots + r_p$ and S is a constant. Moreover $E(X_j) = 0$ and $E(X_j X_k) = M(1, 1) \neq 0$. In this case $E((X_1 + \cdots + X_n)^2) = n^2 M(1, 1) + O(n)$ so that the normalizing factor for the sum is $O(n^{-1})$ rather than $O(n^{-\frac{1}{2}})$. A simple calculation then shows that the moments of the normalized sums approach those of the normal distribution if and only if $M(1, \cdots, 1)$ equals 0 or $(2m)! M^m(1, 1)/(2^m m!)$ according as the number of arguments in $M(1, \cdots, 1)$ is odd or $2m$. A similar statement is true for sequences $\{X_n\}$ satisfying only the weaker condition that $E(X_{i_1} \cdots X_{i_p})$ depends only on p and that $E(X_1^2) + \cdots + E(X_n^2) = O(n)$, and

also for certain slightly more general sequences. A similar theorem is also proved for equivalent sequences with $M(1, 1) = 0$, but under the strong condition that all distributions are odd functions.   *W. Feller* (Ithaca, N. Y.).

Ugaheri, Tadasi.   On a certain sequence of chance variables.   Kōdai Math. Sem. Rep., no. 3, 25–27 (1949).

Let $\{x_n\}$ be a sequence of chance variables with finite expectations satisfying $E_m(x_n) = x_m$ for $m \leq n$, where $E_m(x_n)$ is the conditional expectation of x_n for given $x_1, \cdots, x_m$. Then $\sum_{k=1}^{\infty} 2^{-k} E(|x_{2^{k+1}} - x_{2^k}|) < \infty$ implies

$$\Pr(\lim_{n \to \infty} x_n/n = 0) = 1.$$

A form of the central limit theorem similar to Lindeberg's is also proved for these x_n; this should be compared with a result due to Loève [J. Math. Pures Appl. (9) **24**, 249–318 (1945), in particular, pp. 293–295; these Rev. **7**, 458].   *K. L. Chung* (Ithaca, N. Y.).

MacDonald, D. K. C.   Some statistical properties of random noise.   Proc. Cambridge Philos. Soc. **45**, 368–372 (1949).

When noise is restricted to a relatively narrow range its amplitude can be written in the form $y(t) = R(t) \sin \{\omega_0 t + \theta(t)\}$. The joint distribution of $R(t_1)$, $R(t_2)$, $\theta(t_1)$ and $\theta(t_2)$ was given by Rice [Bell System Tech. J. **23**, 282–332 (1949); these Rev. **6**, 89]. The author derives the probability distribution of $\theta(t_2) - \theta(t_1)$ and discusses some of its properties.   *M. Kac* (Ithaca, N. Y.).

Anzai, Hirotada.   A remark on spectral measures of the flow of Brownian motion.   Osaka Math. J. **1**, 95–97 (1949).

Let f be a function defined on the space of the sample functions of a Brownian motion stochastic process (that is, a process with independent stationary Gaussian increments) and let f_t be the corresponding function of the process after a translation through time t. Then if f is measurable, if $E\{|f|^2\} < \infty$, and if $E\{f\} = 0$, f_t is a random variable for each t, defining a stationary stochastic process with covariance function $E\{f_t f\}$. The covariance function is the Fourier-Stieltjes transform of a monotone function, the spectral distribution function of the f_t process. The author proves that this spectral distribution function is absolutely continuous.   *J. L. Doob* (Ithaca, N. Y.).

Chandrasekhar, S.   Brownian motion, dynamical friction and stellar dynamics.   Dialectica **3**, 114–126 (1949).

The author contrasts the physical theory of Brownian motion, in which the elementary processes of the individual collisions are not discussed, with the corresponding theory of stellar statistics, in which the specific properties of encounters between stars subject to Newtonian forces can be used for example to divide the effects of encounters into two parts, dynamical friction and random fluctuation. The corresponding separation is assumed in Brownian motion theory.   *J. L. Doob* (Ithaca, N. Y.).

*Wiener, Norbert.   Extrapolation, Interpolation, and Smoothing of Stationary Time Series.   With Engineering Applications.   The Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass.; John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1949.   ix+163 pp. $4.00.

This book is essentially a reprint of one that previously had a limited circulation. It contains an exposition of cer-

tain mathematical problems from the point of view of electrical engineering, more specifically from the point of view of communication engineering. In accordance with this point of view the emphasis is on making the mathematics suggestive and on facilitating the realization of certain linear operators by means of electrical apparatus rather than on giving a full and rigorous mathematical treatment. The basic mathematical problem can be described as follows. Let x_t, $-\infty < t < \infty$, be the random variables of a stationary Gaussian stochastic process, with $E\{x_t\} = 0$. The parameter is sometimes integral-valued, sometimes unrestricted; the random variables are sometimes N-dimensional vectors. Let x be a given random variable and let t_0 be fixed. The problem is to minimize $E\{|x-y|^2\}$ for y a linear combination of x_t's with $t \leq t_0$ or a limit in the mean of such linear combinations. For example, if $x = x_s$ (s fixed, $s > t_0$) the problem becomes one of predicting the future value of a time series on the basis of knowledge of the past; with other identifications the problem becomes one of estimating the future value of a time series on the basis of knowledge of the past values of the sum of the given series with another one (filter problem). The solution of this approximation problem has two parts, (A) evaluation of the prediction, (B) evaluation of the mean square error of the prediction. Part (B) includes finding the conditions under which the error vanishes, a condition which has independent interest. Wiener's principal interest is in (A) and in the realization of the best approximation operator by means of electrical circuits. He has described his work in an expository paper [Bol. Soc. Mat. Mexicana 2, 37–42 (1945); these Rev. 7, 461]. Other authors have been principally interested in (B) [see, for example, Kolmogorov, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 3–14 (1941); these Rev. 3, 4; Krein, C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 91–94, 306–309 (1945); these Rev. 7, 156, 61]. The chapters are headed: I, Résumé of fundamental mathematical notions; II, The linear predictor for a single time series; III, The linear filter for a single time series; IV, The linear predictor and filter for multiple time series; V, Miscellaneous problems encompassed by the technique of this book. Finally there are reprinted two papers by Levinson [J. Math. Phys. Mass. Inst. Tech. 25, 261–278 (1946); 26, 110–119 (1947); these Rev. 8, 391; 9, 46] which treat aspects of this same subject. In each problem treated the author shows how to get a minimizing linear operator and discusses the problem of realizing it in practice.   *J. L. Doob.*

Yaglom, A. M.   On problems about the linear interpolation of stationary random sequences and processes. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 173–178 (1949). (Russian)

Let $x(t)$ be the random variables of a stationary (wide sense) process, with $E\{x(t)\} = 0$. Let K be any set of parameter values and let K' be the complement of K. If $s \epsilon K'$ the author considers the problem of approximating $x(s)$ by linear combinations of the $x(t)$'s with $t \epsilon K$ and limits in the mean of such linear combinations. The approximation is to be best in the sense of least squares. The mean square error of the best approximation is evaluated in the following cases: (1) t takes on all real values, K consists of the integers; (2) t is integral-valued, K consists of the odd integers; (3) t is integral-valued, K' consists of n ($< \infty$) integers. The last case was solved for $n=1$ by Kolmogorov [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 3–14 (1941); these Rev. 3, 4].   *J. L. Doob* (Ithaca, N. Y.).

Kolmogorov, A. N., and Prohorov, Yu. V.   On sums of a random number of random terms. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 168–172 (1949). (Russian)

Five theorems are proved, of which we cite two. Theorem 1 is an immediate consequence of theorem 2, and theorems 3 and 5 are special cases of theorem 4. Let ν be a chance variable which takes only nonnegative integral values, $\{\xi_n\}$, $n = 1, 2, \cdots$, an infinite sequence of chance variables, and $\zeta_\nu = \xi_1 + \cdots + \xi_\nu$. Condition w is said to be fulfilled if, for $n > m$, the chance variable ξ_n and the event $\{\nu = m\}$ are independent. Let $p_n = P\{\nu = n\}$ and $P_n = P\{\nu \geq n\}$. Theorem 2 states: if (a) $E|\xi_n| = c_n$ exist for all n, (b) condition w is fulfilled, (c) $\sum_{n=1}^{\infty} P_n c_n < \infty$, then $E\zeta_\nu$ exists and equals $\sum_{n=1}^{\infty} p_n A_n$, where $A_n = \sum_{i=1}^{n} a_i$ and $a_i = E\xi_i$. (Let $c_n^0 = E(|\xi_n| | \nu \geq n)$, $a_n^0 = E(\xi_n | \nu \geq n)$.) [The authors' proof suffices to prove the following somewhat more general theorem: if (a) c_n^0 exist for all n, (b) $\sum_{n=1}^{\infty} P_n c_n^0 < \infty$, then $E\zeta_\nu$ exists and equals $\sum_{n=1}^{\infty} p_n A_n^0$, where $A_n^0 = \sum_{i=1}^{n} a_i^0$. This theorem is very similar to, and is proved in the same way as, theorem 7.4 of a paper by the reviewer [Ann. Math. Statistics 18, 215–230 (1947); these Rev. 9, 49]. This method of proof can be easily extended to higher moments (see below).]

Let $\xi_n = (\xi_n^1, \xi_n^2)$ be a bivariate chance variable. Condition z is said to be fulfilled if the vectors $\xi_1, \xi_2, \cdots$ are mutually independent. Theorem 4 states: if (a) condition w and z are fulfilled, (b) $E\xi_n^i = a_n^i$, $E(\xi_n^i - a_n^i)(\xi_n^j - a_n^j) = b_n^{ij}$, $i, j = 1, 2$, exist for all n, (c) the series $\sum_{n=1}^{\infty} P_n\{(b_n^{11}B_n^{22})^{\frac{1}{2}} + (b_n^{22}B_n^{11})^{\frac{1}{2}}\}$ converges, where $B_n^{ij} = \sum_{i=1}^{n} b_i^{ij}$, then $E(\zeta_\nu^1 - A_\nu^1)(\zeta_\nu^2 - A_\nu^2)$ exists and equals $\sum_{n=1}^{\infty} p_n B_n^{12}$. Here $A_n^i = \sum_{i=1}^{n} a_i^i$. [This theorem can be generalized in the same manner as theorem 2.]   *J. Wolfowitz* (New York, N. Y.).

Kolmogorov, A. N.   A local limit theorem for classical Markov chains. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 281–300 (1949). (Russian)

The author considers a Markov chain with states $\gamma_1, \cdots, \gamma_s$. Let $\mu^{(j)}(t)$ be the number of times the state γ_j is attained in the first t transitions from the initial state γ. Write $\mu(t)$ as the vector with components $\mu^{(j)}(t)$. If $m: (m^{(1)}, \cdots, m^{(s)})$ is a vector with nonnegative integral components, $W^{(\gamma)}(m)$ is the probability that the first $\bar{m} = \sum_j m^{(j)}$ transitions from initial state γ have passed $m^{(j)}$ times through γ_j, $j \leq s$. It is supposed (A) that it is possible to pass from any state to any other by way of transitions with positive probability and (B) that the covariance matrix of $\mu^{(j)}(t)$ has rank $s-1$ as $t \to \infty$, the maximum possible value. It is shown that $\mu(t)$ is then subject to the central limit theorem in the sense that in a certain $s-1$ dimensional space $\mu(t)$, normalized as usual, has a distribution which is asymptotically Gaussian and nondegenerate. The following local limit theorem is also proved:

$$\bar{m}^{(s-1)/2} W^{(\gamma)}(m) = s^{\frac{1}{2}} p(x) + o(1), \quad x = (m - \bar{m}q)/\bar{m}^{\frac{1}{2}},$$
$$E\{\mu(t)\} = qt + O(1),$$

uniformly for bounded $x^{(j)}$. Here $p(x)$ is the normal density involved in the integral form of the central limit theorem. Finally it is shown how these results are to be modified if hypotheses (A) and (B) are not satisfied. Their validity can be checked by a finite procedure involving only an examination of the positions of the nonzero elements of the transition matrix of the process.   *J. L. Doob.*

Consael, R.   Sur quelques processus stochastiques discontinus à deux variables aléatoires. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 399–416 (1949).

Let m_t and n_t be the number of individuals of two species alive at time t. Two sets of hypotheses are made. (I) At

$t=0$ there are M, N individuals of the two types; in time dt each individual changes species with probability $\lambda(t)dt$ and dies with probability $\mu(t)dt$, independently of the past and of the action of other individuals. (II) At $t=0$ there is one individual of one type and none of the other; in time dt each individual induces the appearance of an individual of the other species (without disappearing itself) with probability $\lambda(t)dt$, independently of the past and of the action of other individuals. Under each set of hypotheses the distribution of individuals of the species at time t is found, using generating functions. *J. L. Doob* (Ithaca, N. Y.).

Lévy, Paul. Exemples de processus pseudo-markoviens. C. R. Acad. Sci. Paris **228**, 2004–2006 (1949).

A stochastic process $X(t)$ with a continuous time parameter t is a Markov process if, for any $t_0 < t_1 < \cdots < t_h < t$ and for any $x_0, x_1, \cdots, x_h$, the conditional probability $F(t_0, t_1, \cdots, t_h, t; x_0, x_1, \cdots, x_h, x)$ that $X(t) < x$ under the conditions $X(t_i) = x_i$, $i=0, 1, \cdots, h$ depends only on t_h, t, x_h and x. It is easy to see that if $X(t)$ is a Markov process then the conditional probability $F(t_0, t; x_0, x)$ satisfies the Chapman-Kolmogoroff equation:

$$F(t_0, t; x_0, x) = \int_{-\infty}^{\infty} F(t_1, t; \xi, x) d_\xi F(t_0, t_1; x_0, \xi)$$

for any $t_0 < t_1 < t$ and x_0, x. The author shows, by constructing an example, that the converse is not always true. *S. Kakutani* (New Haven, Conn.).

[Seal, H. L.] Discrete random processes. J. Inst. Actuaries Students' Soc. **8**, 204–209 (1949).

Expository paper. *J. L. Doob* (Ithaca, N. Y.).

TOPOLOGY

Marczewski, Edward. Sur l'isomorphie des relations et l'homéomorphie des espaces. Ann. Soc. Polon. Math. **21** (1948), 336–342 (1949).

If E is any set and R is a binary relation defined in E, write A^- for all those y in E such that yRx for some x in A, where A is any subset of E. The operation $A \to A^-$ is said to be determined by R, and makes E into an "allgemein-topologischen Raum" (ATR). As to the converse, the operation $A \to A^-$ in an ATR is determined by a relation R if and only if for $x \varepsilon A^-$ there exists $y \varepsilon A$ such that $x \varepsilon \{y\}^-$. The ATR determined by R is a T-space if and only if R is symmetric and reflexive; and it is a T_1-space if and only if R is the identity relation, whereupon E becomes discrete. *R. Arens* (Los Angeles, Calif.).

Kuratowski, Casimir. Sur la notion de limite topologique d'ensembles. Ann. Soc. Polon. Math. **21** (1948), 219–225 (1949).

For spaces X in which a notion of convergent sequence is defined one can adopt the wording of the usual definition of limit inferior ($\mathrm{Li}_n A_n$) and limit superior ($\mathrm{Ls}_n A_n$) of a sequence A_n of subsets of X. Writing (*) $\mathrm{Lim}_n A_n = A$ if $A = \mathrm{Li}_n A_n = \mathrm{Ls}_n A_n$ one obtains $F = (2^X)_L$, the class of subsets of X endowed with the notion (*) of convergent sequences. The properties of F when X is compact metric are fairly well known. It is desirable to develop the theory of F entirely within the theory of (some suitable) convergence spaces. This paper shows that this is possible, and that there is a type of space [the L^*-space; see the author's Topologie I, 1st ed., Warszawa-Lwów, 1933, p. 76] for which the theory assumes a natural form. The results include the following. (1) If X is an L^*-space in which $A^{--} = A^-$ (i.e., a topological L^*-space) then F is an L^*-space, although $S^{--} = S^-$ may fail in F. (2) If Y and Z are closed disjoint subsets of a topological L^*-space X then $(2^{Y+Z})_L$ is homeomorphic with the Cartesian product of $(2^Y)_L$ and $(2^Z)_L$. *R. Arens* (Los Angeles, Calif.).

Sierpiński, Wacław. Sur la décomposition des espaces métriques en ensembles disjoints. Fund. Math. **36**, 68–71 (1949).

Let M be a metric space such that every (nonempty) open set in M contains at least $\mathfrak{m} \geq \aleph_0$ points. Then M is the sum of $\mathfrak{m}$ disjoint sets each of which contains at least $\mathfrak{m}$ points of each open subset of M. This is the solution, for the metric case, of the problem of "determining the largest number of disjoint dense subsets possible in a resolvable space," proposed by E. Hewitt [Duke Math. J. **10**, 309–333 (1943); these Rev. **5**, 46; a space is resolvable if it is the sum of two disjoint dense sets]. For $\mathfrak{m} = \aleph_0$, it reduces to the statement that every metric space M which is dense in itself is the sum of an infinite sequence of disjoint dense sets. A set is condensed if each of its open subsets is uncountable. The theorem given above implies that every condensed metric space is a sum of uncountably many disjoint condensed sets each of which is dense in the space. *E. E. Moise* (Princeton, N. J.).

Gołąb, St. Espace pourvu d'une métrique définie au moyen de l'écart triangulaire et les espaces métriques généralisés. Ann. Soc. Polon. Math. **21** (1948), 226–235 (1949).

In a convex metric space, let $\rho(p, q)$ be a secondary distance satisfying only the following conditions: (1) $\rho(p, p) = 0$; (2) $\rho(p, q) > 0$ if $p \neq q$; (3) if, in the metric space, the points p_n and q_n converge to p and q, respectively, then $\lim \rho(p_n, q_n) = \rho(p, q)$. Let $I(p, q)$ denote the set of all points x such that $\rho(p, x) + \rho(x, q) \leq \rho(p, q)$. If all the sets $I(p, q)$, for the various pairs p, q, have a nonvacuous intersection, then every two points can be joined by a geodesic arc of length 0. This theorem is applicable to the Euclidean plane with the secondary distance $\rho(p, q) = |p_1 q_2 - q_1 p_2|$ if (x_1, x_2) are the Cartesian coordinates of the point x. For, the origin is common to all sets $I(p, q)$. *K. Menger* (Chicago, Ill.).

Haupt, Otto. Schwach ordnungsminimale Kontinua im projektiven R_n. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 75–76 (1949).

The author considers a continuum K in the n-dimensional projective space R_n. The order value of K is by definition the maximum number of points (if a maximum exists) in $K \cdot R_{n-1}$, where R_{n-1} ranges over the set of hyperplanes in R_n. If R_{n-1} ranges over all hyperplanes except for a nowhere dense set, then the corresponding maximum is called the weak order value. Several theorems are stated without proof. For example, every continuum in R_n with finite weak order value is a hereditary arc-sum; for continua K in R_n having weak order value equal to n, the number of junction points has a finite upper bound depending only on n. For earlier work along this line see Haupt [Ann. Mat. Pura Appl. (4) **23**, 123–148 (1944); these Rev. **7**, 468]. *J. H. Roberts* (Durham, N. C.).

Sitnikov, K. On some metric properties of closed sets. Doklady Akad. Nauk SSSR (N.S.) **67**, 229–232 (1949). (Russian)

The author solves a problem posed by P. Alexandroff [Ann. of Math. (2) **30**, 101–187 (1928)]. If A is a finite set of points in a metric space R, and $x \epsilon R$, then the order of A at x is the number of points of A which are all at the same minimal distance from x. The order of A in R is the largest such order, for all x in R. An n-dimensional compactum F is called metrically correct if for every $\epsilon > 0$ it contains a finite ϵ-dense set (ϵ-net) which is of order $n+1$ in F. It is proved that every polyhedron is correct and that every n-dimensional compactum F is homeomorphic to some metrically correct compactum F^*. If F lies in R^{2n+1}, then the F^* can be found arbitrarily "close" to F.
L. *Zippin* (Flushing, N. Y.).

Moise, Edwin E. A theorem on monotone interior transformations. Bull. Amer. Math. Soc. **55**, 810–811 (1949).

The following theorem is proved. For no compact, metric, irreducible continuum M is there a monotone interior transformation throwing M into an arc A such that the inverse image of each point of A is an arc. This answers a question raised by Knaster [Fund. Math. **25**, 568–577 (1935)].
J. H. *Roberts* (Durham, N. C.).

Liao, S. D. On non-compact absolute neighbourhood retracts. Acad. Sinica Science Record **2**, 249–262 (1949).

The author proves the known generalization of a theorem of Lefschetz [Topics in Topology, Princeton University Press, 1942, theorem 6.2, p. 82; these Rev. **4**, 86] to arbitrary separable metric spaces: LC^n (LC^∞) is equivalent with the property that any covering σ has a refinement σ' such that any partial realization of a polytope P dim $P \leq n+1$ (dim P arbitrary) in σ' extends to a full realization in σ. The paper also contains a direct proof of the fact that every locally finite polytope is an absolute neighborhood retract. [Reviewer's note: the basic results on compact LC^n and ANR spaces can in fact be extended to arbitrary metric spaces, by using the paracompactness of metric spaces.] J. *Dugundji* (Los Angeles, Calif.).

Vaĭnšteĭn, I. A. On dimension-increasing mappings. Doklady Akad. Nauk SSSR (N.S.) **67**, 9–12 (1949). (Russian)

Let $f: X \to Y$ denote a closed (continuous) mapping, X and Y being separable metric spaces. The cardinal power of the set $f^{-1}(y)$, for each $y \epsilon Y$, is called the multiplicity (of f) at y, and denoted by $\mu(y)$. A subset L of $f^{-1}(y)$, $L \subset X$, is called correct (with respect to f) if: U open in X, $U \supset L$ implies that $f(U)$ contains y as inner point. For closed mapping f, $f^{-1}(y)$ is itself a correct set. The smallest integer k for which there is a correct set L of k points is called the order of f at y, denoted by $\nu(y)$. This order is taken as infinite, if no k exists. For open mappings, $\nu(y) = 1$. For closed mappings, $\nu(y) \leq \mu(y)$. Let Y_k denote the set of $y \epsilon Y$ such that $\mu(y) \geq k$, Y_k' that for which $\nu(y) \geq k$. Then f is said to have property A if it is zero-dimensional and for every y, either $f^{-1}(y)$ is somewhere dense in X or $f^{-1}(y)$ has an isolated point. All countably-valued mappings have property A, since $f^{-1}(y)$ is closed and countable, and must have an isolated point. The principal theorem of the note is as follows: let $f: X \to Y$ be a closed mapping, having property A. Then for any k, from the premiss dim $Y_k' > \dim X$, it follows that dim $Y_{k+1}' \geq \dim Y_k' - 1$. The proof distin-

guishes the cases $k = 1$ and $k \geq 2$. Among the consequences of this theorem are stated: (1) if f is closed and finite-valued and dim $Y = \dim X + n$, then the order $\nu(y)$ takes on at least $n+1$ different values; (2) if f is closed, open, and countably-valued, then dim $Y = \dim X$. L. *Zippin*.

Každan, Ya. M. On continuous mappings which increase dimension. Doklady Akad. Nauk SSSR (N.S.) **67**, 19–22 (1949). (Russian)

With definitions of order and multiplicity as in the preceding review, one has the following results. (1) Let f be a continuous mapping of an F_σ-space X (i.e., a space representable as the sum of a countable number of compacta) onto a space Y. Let dim $Y - \dim X = n$. If the inverse image $f^{-1}(y)$ of every point $y \epsilon Y$ whose multiplicity is not less than n contains at least n isolated points of the set $f^{-1}(y)$, then in the space Y there exists at least one point y whose order is at least $n+1$. A space X is said to have property α if every nowhere-dense subset of X is of lower dimension. (2) Let X be an F_σ-space with property α, f a continuous mapping of X on Y, and dim $Y - \dim X = n$. If there are always at least $n+1$ isolated points in $f^{-1}(y)$ for every $y \epsilon Y$ for which $\mu(y) \geq n+1$, then there is at least one point $y^* \epsilon Y$ whose order is at least $n+2$. If f is a continuous map of a zero-dimensional X on the n-dimensional cube I_n, then there must exist in I_n at least one point of order at least $n+1$. (4) If f is continuous on the one-dimensional compactum X to the n-dimensional cube I_n, and if f has property α, then there is at least one point in I_n of order at least $n-1$. (5) Let every point x of the compactum L have a countable Urysohn index, and let f be a continuous map of L on the n-dimensional cube I_n. Then there is a point whose order is at least n. All of these theorems are proved. L. *Zippin* (Flushing, N. Y.).

Borsuk, Karol. On topological approximation of polytopes. Ann. Soc. Polon. Math. **21** (1948), 257–276 (1949).

Since no purely topological characterization is known of those spaces which are homeomorphic to polytopes (= finite geometric complexes), the author wishes to define classes of spaces which approximate these spaces. Moreover, he wishes his definitions to be topological in the following sense. Let τ denote the class of all polytopes and let K be a class of mappings which is transitive, i.e., if f, g are in K, with the range of f equal to the domain of g, then gf is also in K. We denote by $K(\tau)$ the class of all spaces $f(P)$ where f runs through K and P runs through τ. Then $K(\tau)$ is called an approximation to the class of polytopes. It is said to be a topological approximation if both the class K of mappings and the class $K(\tau)$ of spaces can be defined in a purely topological way. Thus, for example, $K_1(\tau)$ is such a topological approximation, where K_1 is the class of all mappings, for K_1 is transitive and $K_1(\tau)$ consists of all locally connected compacta. On the other hand, $K_\infty(\tau)$, where K_∞ is the class of all homeomorphisms, is not known to be topological.

In this paper the following are shown to be topological approximations to the class of all polytopes. (1) The class $K_2(\tau)$, where K_2 consists of all r-continuous mappings, i.e., those of the form hr where r is a retraction and h is a homeomorphism. Here $K_2(\tau)$ consists of all finite-dimensional locally contractible compacta. (2) The class $K_{2\Delta}(\tau)$, where $K_{2\Delta}$ is a suitably defined subset of K_2, and the class $K_{2\Delta}(\tau)$ consists of the class of finite-dimensional compacta having property Δ, i.e., for each neighborhood U of a point, there is a smaller neighborhood V such that each compact set E in V is homotopic to a point on a compact

set F in U with dim $F \leqq 1+$ dim E. (3) The class $K_{2A}(\pi)$, consisting of all finite-dimensional compacta which can be decomposed into a finite set of arbitrarily small pieces, any nonvacuous intersection of which is an absolute retract. Here K_{2A} is another suitably defined subset of K_2. (4) The class $K_3(\pi)$, where $K_3 = K_{2A} \cap K_{2A}$. It is known that $K_3(\pi) \subset K_{2A}(\pi) \subset K_2(\pi) \subset K_1(\pi)$ and $K_{2A}(\pi) \subset K_2(\pi)$, all of these being proper inclusions. It is not known whether or not $K_3(\pi) = K_{2A}(\pi)$.   *E. G. Begle* (New Haven, Conn.).

Steenrod, N. E. Cohomology invariants of mappings. Ann. of Math. (2) **50**, 954–988 (1949).

This is the detailed account of results announced earlier [Proc. Nat. Acad. Sci. U. S. A. **33**, 124–128 (1947); these Rev. **8**, 525] concerning certain homotopy invariants of maps, which generalize the well-known Hopf invariant. Let $f: X' \to X$ be a map, $_1H$, $_2H$ two cohomology theories paired to a third H by the cup product. Take $u \epsilon_1 H^p(X)$, $v \epsilon_2 H^q(X)$ satisfying $u \cup v = 0$, $f^*u = 0$. A homotopy invariant $u \cup_f v$ (denoted by $[f, u, v]$ in the author's earlier notation), called the functional cup product, is then defined, which is an element of $H^{p+q-1}(X')/f^*H^{p+q-1}(X') + _1H^{p-1}(X') \cup f^*v$. There is also a similar invariant $u \cap_f v$, called the functional cap product. Proofs of their properties are given by means of the axiomatic homology theory and the notion of mapping cylinder. Applied to orientable manifolds X with a regular boundary A, with f interpreted as the inclusion map $i: A \to X$, the Poincaré-Lefschetz duality theorem yields a theorem to the effect that certain pairings defined by $\cup_i$ are completely orthogonal. This theorem is applied to deduce the central theorem in Gysin's work on sphere bundles, which the author acknowledges to be the motive leading to the present study. Let X denote a compact orientable manifold without boundary and X' an orientable sphere bundle with base space X and fibres of dimension $k > 0$. The mapping cylinder X_f of the projection $f: X' \to X$ is fibred into $(k+1)$-cells and is a bundle over X with projection $f': X_f \to X$. Hence X_f is an orientable $(n+k+1)$-manifold having X' as regular boundary. An application of the Poincaré-Lefschetz theorem then yields Gysin's theorem, which can be stated as follows. Let z' be a generator of the cyclic group $_2H^{n+k}(X')$, and let $_1K^q(X)$ be the kernel of $f^*: _1H^q(X) \to _1H^q(X')$. Then the operation which sends $v \epsilon_1 K^q(X)$ into $v \cap_f z'$ is an isomorphism of $_1K^q(X)$ onto $H_r(X)/f^*H_r(X')$, $r = n+k-q+1$.

The operation is extended to the squaring operation of the author, leading to a functional squaring operation. For a sphere mapping $S^{n+\alpha-1} \to S^n$ it gives a generalized Hopf invariant $\gamma_\alpha(f)$. For $\alpha = 2$ this is the only homotopy invariant in the sense that two maps f, g are homotopic if and only if $\gamma_\alpha(f) = \gamma_\alpha(g)$. The author remarks that this invariant is theoretically calculable.   *S. Chern* (Chicago, Ill.).

Pontryagin, L. S. The homotopy group $\pi^{n+1}(K_n)$ ($n \geqslant 2$) of dimension $n+1$ of a connected finite polyhedron K_n of arbitrary dimension, whose fundamental group and Betti groups of dimensions $2, \cdots, n-1$ are trivial. Doklady Akad. Nauk SSSR (N.S.) **65**, 797–800 (1949). (Russian)

Let K_n ($n > 1$) be a finite connected polyhedron whose homotopy groups $\pi_q(K_n)$ are trivial for $q = 1, \cdots, n-1$, and let $\Phi^r: \pi_r(K_n) \to H_r(K_n)$ ($r = 1, 2, \cdots$) denote the natural homomorphism of the homotopy groups into the integral homology groups. According to a well-known theorem of Hurewicz, Φ^n is an isomorphism onto. It is also known that Φ^{n+1} is a homomorphism onto; let $\pi^0_{n+1}(K_n)$ denote the kernel of Φ^{n+1}. The author determines generators and rela-

tions for $\pi^0_{n+1}(K_n)$. In case $n = 2$, this requires a knowledge of the homology and cohomology groups and cup products with integers and integers mod m ($m = 2, 3, \cdots$) for coefficients and the "Pontrjagin squares" introduced by the author for this purpose in a previous note [C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 35–37 (1942); these Rev. **4**, 249; cf. also J. H. C. Whitehead, Comment. Math. Helv. **22**, 48–92 (1949); these Rev. **10**, 559].

For the case $n > 2$, a knowledge of the homology and cohomology groups with integers and integers mod 2 for coefficients and the cup-i products of Steenrod [Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154] for $i = n-2$ is required. The result for this case has been obtained independently by J. H. C. Whitehead [Ann. Soc. Polon. Math. **21** (1948), 176–186 (1949); these Rev. **11**, 48]. G. W. Whitehead has also obtained an analogous result for general spaces [Proc. Nat. Acad. Sci. U. S. A. **34**, 207–211 (1948); these Rev. **10**, 392]. These results show that $\pi_{n+1}(K_n)$ is a group extension of a known group $\pi^0_{n+1}(K_n)$ by $H_{n+1}(K_n)$. The author then states a theorem on how this group extension may be effectively determined, and thus shows that $\pi_{n+1}(K_n)$ may be effectively calculated in terms of the invariants of K_n listed above.

A method is given for associating with any simplicial map $f: S^{n+1} \to S^n$ an integer in case $n = 2$ or an integer mod 2 in case $n > 2$ which is an invariant of the homotopy class of f. For $n = 2$ this turns out to be equal to the Hopf invariant of f. [Reviewers' note: such an invariant has also been defined by N. E. Steenrod; see the preceding review.] The last theorem gives a necessary and sufficient condition that an element of $H_{n+2}(K_n)$ be a spherical homology class, i.e., belong to the image subgroup of Φ^{n+2}. No proofs are given in this note.   *R. H. Fox and W. S. Massey.*

Pontryagin, L. S. On a connection between homology and homotopy. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 193–200 (1949). (Russian)

This paper is devoted to the proof of the following result, which was stated as a lemma without proof in a previous note by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 35–37 (1942); these Rev. **4**, 249]. Let $\varphi: L \to K$ be a continuous map of an n-dimensional polyhedron L into a connected, simply connected, polyhedron K. Then there exists a map $\psi: L \to K$ homotopic to φ such that $\psi(L)$ is contained in the $(n-1)$-dimensional skeleton of K if and only if the induced homomorphisms $\varphi_*: H_n(L) \to H_n(K)$ are trivial with integers and integers mod m ($m = 2, 3, \cdots$) for coefficients of the homology groups H_n. The proof breaks up into two cases, depending on whether $n > 2$ or $n = 2$. The author observes that if the homomorphism $\varphi_*: H_{n-1}(L) \to H_{n-1}(K)$ is also trivial for the above listed coefficient groups, we cannot conclude that φ is homotopic to a map ψ of L into the $(n-2)$-skeleton of K. Counterexample: the Hopf map of S^3 on S^2, $n = 3$. Also, the condition that K be simply connected cannot be omitted as is shown by the example $L = S^n$, $K = $ real projective n-space (n even), φ the two-fold covering map.

        R. H. Fox and W. S. Massey (Princeton, N. J.).

Komatu, Atuo. Relations between homotopy and homology. I. Osaka Math. J. **1**, 150–155 (1949).

Let K be a connected complex and K^n the n-skeleton of K. In case K is simply connected, the author notes that the relative homotopy group $\pi_n(K^n, K^{n-1})$ is isomorphic for $n > 2$ to the group $C_n(K)$ of n-chains with integral

coefficients. Furthermore, the homology boundary operator $\partial : C_n(K) \to C_{n-1}(K)$ is equivalent to the composition of the homotopy boundary operator $\partial_i : \pi_n(K^n, K^{n-1}) \to \pi_{n-1}(K^{n-1})$, and the injection homomorphism

$$r : \pi_{n-1}(K^{n-1}) \to \pi_{n-1}(K^{n-1}, K^{n-2}).$$

By combining these facts with the exactness of the homotopy sequences of the pairs (K^{n+1}, K^n), (K^n, K^{n-1}), etc., some relations between the homology groups of K, the group of spherical homology classes of K, and various homotopy groups of K, K^n, K^{n-1}, etc., are obtained. The case where K is not simply connected is also considered and by using an analogous type of reasoning, some relations are derived which involve the homology and homotopy groups of K and $\bar{K}$, the universal covering complex of K. The author notes that many of his results have been previously obtained by G. W. Whitehead and others. *W. S. Massey.*

Kudo, Tatsuji. Contribution to the problem of stability. Osaka Math. J. 1, 62–72 (1949).

Se référant à la notion de stabilité introduite par H. Hopf et E. Pannwitz [Math. Ann. **108**, 433–465 (1933)] (un complexe K^n est "stable" si $f(K^n) = K^n$ pour toute application continue f homotope à l'identité), l'auteur donne une démonstration d'une conjecture de Komatu, qui généralise les théorèmes de Hopf-Pannwitz, et se formule ainsi : pour qu'un complexe K^n ($n \neq 2$), localement fini et homogène, soit stable, il faut et il suffit qu'il soit cyclique, c'est-à-dire que, pour tout n-simplexe σ_n, existent un système local de coefficients et un n-cycle (relatif à ce système local) tel que le coefficient de σ_n soit $\neq 0$ dans ce cycle. Le cas où $n = 2$ reste ouvert. *H. Cartan* (Paris).

Hu, Sze-tsen. On joins of spherical mappings. Fund. Math. 36, 23–34 (1949).

G. W. Whitehead has defined [Proc. Nat. Acad. Sci. U. S. A. **32**, 188–190 (1946); these Rev. **8**, 50] an operation which associates with elements $\alpha \epsilon \pi_p(S^m)$ and $\beta \epsilon \pi_q(S^n)$ an element $\alpha * \beta \epsilon \pi_{p+q+1}(S^{m+n+1})$, called the join of α and β. In this paper several properties of this join operation are proved, the principal ones being the following. (1) The join operation is bilinear; i.e., $(\alpha + \alpha') * \beta = (\alpha * \beta) + (\alpha' * \beta)$, and $\alpha * (\beta + \beta') = (\alpha * \beta) + (\alpha * \beta')$. (2) The join operation is associative: $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$. (3) If either α or β is a Whitehead product, then $\alpha * \beta = 0$. *W. S. Massey* (Princeton, N. J.).

Morse, Marston. L-S-homotopy classes of locally simple curves. Ann. Soc. Polon. Math. 21 (1948), 236–256 (1949).

The concepts of locally simple curve and angular order of a plane curve have been introduced by Morse and Heins [cf. Proc. Nat. Acad. Sci. U. S. A. **31**, 299–301, 302–306 (1945); these Rev. **7**, 57]. The L-S-homotopy classes are defined by admitting only deformations for which the deformed curves are in a sense uniformly locally simple. The present paper is concerned with locally simple curves on a closed two-dimensional orientable manifold of finite genus. The author extends the definition of angular order to the case under consideration, there being three essentially different cases according as S is a sphere, a torus, or a surface of genus at least 2. A set M_S of curves on S is a model for the L-S-homotopy classes if no two members of M_S are L-S-homotopic and any sensed locally simple curve is L-S-homotopic to some member of M_S. Models for the L-S-homotopy classes are displayed for all cases, as well as topological invariants numerically characteristic of these classes. In the case of a sphere, an arbitrary sensed simple closed curve and the same curve taken twice form a model, and the angular order is, correspondingly, 1 or 2. Corresponding but more complex results are obtained in the other cases. *G. A. Hedlund* (New Haven, Conn.).

Freudenthal, H. Note on the homotopy groups of spheres. Quart. J. Math., Oxford Ser. 20, 62–64 (1949).

Let $f : S^e \to S^d$ be a continuous map of an e-dimensional sphere onto a d-dimensional sphere ($d < e$) which is simplicial with respect to certain triangulations of S^e and S^d. Let p be an interior point of any d-simplex of this triangulation of S^d. Then it is well known that $f^{-1}(p)$ is a pseudo-manifold of dimension $e - d$, which may have several components. In a previous paper [Compositio Math. **5**, 299–314 (1937)], the author asserted that f is always homotopic to a map $g : S^e \to S^d$ having the property that $g^{-1}(p)$ is a connected pseudo-manifold. The proof given there contained an error; this paper contains a correct proof. *W. S. Massey.*

GEOMETRY

Jaśkowski, S. Une modification des définitions fondamentales de la géométrie des corps de M. A. Tarski. Ann. Soc. Polon. Math. 21 (1948), 298–301 (1949).

The author suggests a modification of the definition given by Tarski [Księga Pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego, Lwów, 1927, pp. 29–33 (1929)] for the relation, "the sphere A is concentric with the sphere B," occurring in a postulational system in which sphere is primitive. The note defines (1) sphere A to be a saturated subsphere of a body B whenever $A \subset B$, and $A \subset X \subset B$ (X, a sphere) implies $A = X$, while (2) A is a concentric subsphere of sphere B provided $A \subset B$, and X, Y, Z being spheres with Z a saturated subsphere of $A' \cdot Y'$, then $A \subset X \subset Z'$, $Z \subset B \subset Y'$ imply $X \cdot Y = 0$. (The symbols $\subset$, $\cdot$, $'$ denote inclusion, product, and complement, respectively, in a Boolean algebra). It is stated that definition 2 may take the place of Tarski's definition of concentric spheres. *L. M. Blumenthal* (Columbia, Mo.).

Schiek, Helmut. Mengen mit affiner Anordnung. Arch. Math. 1, 473–479 (1949).

Let G be a set with a between relation satisfying the classical postulates which guarantee the existence of segments, lines, simplexes, etc. A subset L of G is called a linear space if L contains every line joining any two distinct points of L. Let $T(U)$ denote the smallest linear space containing a given subset U of G. Then, if P is a point of $T(U)$, there exist two disjoint finite subsets U_1 and U_2 of U such that P lies on the line joining two points of the simplexes determined by U_1 and U_2. For every linear space L, there exists a base, i.e., a set U such that $L = T(U)$ but $L \neq T(U')$ for any proper subset U' of U. Any two bases of L have the same power $D(L)$, and $D(L_1) + D(L_2) = D(L_1 + L_2) + D(L_1 \cdot L_2)$. If we call M open in L if, for each point P of M, every line through P has a segment with M in common, then the convex open sets make L a T_1-space. *K. Menger* (Chicago, Ill.).

✳**Efimov, N. V. Vysšaya Geometriya.** [Higher Geometry]. 2d ed. OGIZ, Moscow-Leningrad, 1949. 502 pp. The first edition appeared in 1945; these Rev. **7**, 256.

✳**Sperner, Emanuel. Grundlagen der Geometrie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 113–132. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Andreotti, Aldo. Sulla proposizione di De Zolt pei poliedri. Boll. Un. Mat. Ital. (3) **4**, 68–75 (1949).

Ordnet man einem Tetraeder das Produkt aus dem Inhalt einer Seitenfläche und der zugeordneten Höhe als charakteristische Zahl zu, so ist diese Zuordnung, ohne sich auf das Archimedische Postulat zu berufen, von der Wahl der Seitenfläche unabhängig. Es wird in etwas anderer Form als bei Veronese [Atti Ist. Veneto Sci. Lett. Arti (7) **6**(53), 421–437 (1895)] bewiesen, dass bei jeder Zerlegung eines Tetraeders in Teiltetraeder die charakteristische Zahl des Gesamttetraeders gleich der Summe der charakteristischen Zahlen der Teiltetraeder ist. Man schliesst dann weiter, dass die Polyederkörper eine Grössenklasse bilden, und dass für sie das Theorem von de Zolt gilt, nach dem ein in Teilpolyeder zerlegtes Polyeder nach Entfernung eines Teilpolyeders den Rest weder zerlegungs- noch ergänzungsgleich ist. *R. Moufang* (Frankfurt am Main).

Hjelmslev, Johannes. Einleitung in die allgemeine Kongruenzlehre. VI. Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 10, 27 pp. (1949).

This paper is a continuation of the third and fifth communications in this series [same Medd. **19**, no. 12 (1942); **22**, no. 13 (1945); these Rev. **7**, 472; **8**, 83]. Like the cited papers, it deals with a plane geometry in which two points can always be connected by a straight line, but the intersection of several straight lines may be a whole segment ("osculating straight lines"). Congruence is defined by means of a group C of incidence-preserving transformations, which is transitive both with respect to the points and the straight lines. To every straight line a there is one and only one transformation in C different from the identity which leaves all the points of a fixed; it has no other fixed points ("reflection a"). The reflections serve to define the orthogonality of two straight lines and to introduce the midpoints of segments.

Let a and b be two straight lines intersecting at a point O. Suppose a and b are not almost orthogonal, i.e., b and the normal of a at O are neither identical nor osculating. By mapping each point P on the midpoint of the segment connecting P with its image under the "rotation" ab, we obtain the "half-rotation" (a, b) about O. The half-rotations about O and their inverses generate a commutative group S. If two nonosculating straight lines a and b intersect at O, then abc is an involution if and only if O lies on c; the straight lines through O are said to form a real pencil. Let a and b be two nonintersecting straight lines such that the straight lines connecting the points of a with those of b are always uniquely determined. Then the set of all the straight lines c such that abc is an involution forms an "ideal pencil." Through each point there exists exactly one straight line of an ideal pencil. To each pencil of this kind we associate an ideal point that lies on all the straight lines of the pencil. The point associated to the ideal pencil of all the normals of a straight line is called the pole of that line. Let O be a given point ("fundamental point"). If any ideal pencil π is given, let a be its straight line through O. The pencil π and its

ideal point are called regular if π contains a straight line b whose normal a' through O is not almost orthogonal to a. In that case any three straight lines of π are in involution and the half-rotation (a, a') transforms π into a real pencil.

The "enlarged plane" Ω of all the real and the regular ideal points is invariant under S. The same holds true of the "fundamental straight line" ω, i.e., the set of the poles of all the straight lines through O. A "regular ideal straight line" is transformed into a real straight line through a suitable half-rotation about O. Thus the set of the real and of the regular ideal straight lines is also invariant under S. By means of ω and of S various concepts can be defined in or extended to Ω: parallelism, angles and their equality, similarity, orthogonality, reflection, and congruence. The group C of all congruences can also be enlarged and Ω becomes a plane "of Euclidean type" in which rectangular coordinate systems can be introduced. The coordinates are elements of a ring in which every sum of nonzero squares is a nonzero square. The straight lines are represented by linear equations. *P. Scherk* (Saskatoon, Sask.).

Gál, I. S. A theorem concerning closed polygons. Ann. Mat. Pura. Appl. (4) **27**, 261–265 (1948).

Let $S(P)$ denote the sum of the distances from the point P to the sides of a polygon. The author proves the following theorem: $S(P)$ is constant if and only if the sides of the polygon are parallel to those of an equilateral polygon. To the paper is added a short proof by B. Jessen.
O. Bottema (Delft).

✳**Hermann, Carl. Kristallgeometrie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 59–62. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

✳**Süss, Wilhelm. Analytische und höhere Geometrie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 133–147. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

✳**Zacharias, Max. Elementarmathematik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 23–38. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Zacharias, Max. Neue Wege zur Hesseschen Konfiguration $(12_4, 16_3)$. Math. Nachr. **2**, 163–170 (1949).

The reciprocal of the n-dimensional hypercube is the cross polytope β_n, whose vertices have for Cartesian coordinates the permutations of $(\pm 2, 0, 0, 0, \cdots)$. The midpoints of the edges, namely $(\pm 1, \pm 1, 0, 0, \cdots)$, etc., are the $2n(n-1)$ vertices of the truncated cross polytope $t_1\beta_n$ [Coxeter, Philos. Trans. Roy. Soc. London. Ser. A. **229**, 329–425 (1930), in particular, pp. 354–360], which has $4n(n-1)(n-2)$ edges belonging by sixes to $\frac{2}{3}n(n-1)(n-2)$ regular hexagons lying in planes through the center (since the edge-length of $t_1\beta_n$ is equal to its circumradius). Thus the $n(n-1)$ "diameters" of $t_1\beta_n$ lie by threes in $\frac{1}{3}n(n-1)(n-2)$ planes, and the section by an arbitrary hyperplane is an $(n-1)$-dimensional configuration of $n(n-1)$ points lying by threes on $\frac{2}{3}n(n-1)(n-2)$ lines. By projection we obtain a plane configuration

$$[n(n-1)_{2(n-2)}, \ \tfrac{2}{3}n(n-1)(n-2)_3],$$

which the author calls the Ceva-Menelaus derivative of the complete n-point $A_1A_2\cdots A_n$. The above coordinates for

$t_i\beta_n$ show that the $n(n-1)$ points of this plane configuration are $A_i \pm A_j$ ($i<j$) in the notation of Möbius's barycentric calculus. When $n=4$, $t_i\beta_n$ is the regular 24-cell $\{3, 4, 3\}$; therefore the $(n-1)$-dimensional configuration is in this case Stephanos's "desmic" 12_8 [E. Hess, Abh. Leopold.-Carolin. Akad. Nat. **75** (1899), p. 170], which projects into L. O. Hesse's $(12_4, 16_3)$ by a previous result of the author [Deutsche Math. **6**, 147–170 (1941), in particular, p. 152; these Rev. **8**, 219]. Hence Hesse's configuration is the same $(12_4, 16_3)$ as the Ceva-Menelaus derivative of the complete quadrangle. *H. S. M. Coxeter* (Toronto, Ont.).

Rédei, L., und Sz.-Nagy, B. Eine Verallgemeinerung der Inhaltsformel von Heron. Publ. Math. Debrecen **1**, 42–50 (1949).

The following theorem is proved in an elementary manner. Let I and J denote the areas of two polygons $A_1A_2 \cdots A_m$ and $B_1B_2 \cdots B_n$ in the Euclidean plane. Then

$$16IJ = \sum_{i=1}^{m} \sum_{k=1}^{n} \{(A_iB_k)^2(A_{i+1}B_{k+1})^2 - (A_iB_{k+1})^2(A_{i+1}B_k)^2\}.$$

In particular,

$$16I^2 = \sum_{i=1}^{m} \sum_{k=1}^{m} \{(A_iA_k)^2(A_{i+1}A_{k+1})^2 - (A_iA_{k+1})^2(A_{i+1}A_k)^2\}.$$

Moreover, these expressions are unique, in the sense that they cannot be replaced by any other polynomial functions of the distances A_iB_k or A_iA_k, respectively. The theorem is extended from polygons to closed curves. Let r denote the distance from the point with parameter u on one curve to the point with parameter v on the other; then the product of the areas is given by

$$8IJ = \int \int r^2\left(r\frac{\partial^2 r}{\partial u \partial v} - \frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\right)du\,dv,$$

the range of values of u and v being such as to take each point all round its curve. *H. S. M. Coxeter.*

Sz.-Nagy, Gyula. Gleichseitige Hyperbel und Parallelogramme. Publ. Math. Debrecen **1**, 24–28 (1949).

With two given fixed segments AA', BB' the author associates the locus of a point P satisfying the relations $PA \cdot PA' = PB \cdot PB'$. This locus is (a) an equilateral hyperbola or (b) a circular cubic depending upon whether the mid-points of the two segments are coincident or not. Special cases of (a) are considered, as well as the converse of that proposition. Using complex coordinates the author shows that at any point of an equilateral hyperbola two equal and parallel chords of that curve subtend angles which are either equal or supplementary. *N. A. Court.*

Sypták, M. Hypercirconférences et hyperhélices. Publ. Fac. Sci. Univ. Masaryk no. **312**, 41 pp. (1949). (Czech. French summary)

L'auteur s'occupe de l'étude des courbes de l'espace euclidien à p dimensions R_p ($p \geqq 2$) dont toutes les courbures scalaires sont constantes. Ces courbes sont appelées hypercirconférences si p ($\geqq 4$) est pair et hyperhélices si p ($\geqq 5$) est impair. Dans le premier chapitre figurent les formules préliminaires. Dans le chapitre II sont déduites les équations des hypercirconférences et celles des hyperhélices. Le chapitre III contient les propriétés des courbes en question. Dans le chapitre IV sont traitées les conditions nécessaires et suffisantes pour qu'une courbe de l'espace R_p jouisse de la propriété d'avoir toutes les courbures scalaires constantes.

On peut trouver un résumé plus détaillé dans les C. R. Acad. Sci. Paris **195**, 298–299 (1932).
From the author's summary.

Gambier, Bertrand. Trisectrices des angles d'un triangle. Mathesis **58**, 174–215 (1949).

This is a detailed and careful study of the celebrated Morley problem. Ten figures accompany the exposition. The author observes that the form of the figure consisting of the nine lines which contain the foci of the variable cardioid tangent to the sides of the triangle ABC is topologically always the same; the precise relative position of the triangle ABC with respect to those nine lines depends upon whether the triangle is obtuse-angled or not.

To the comprehensive bibliography of the author may be added the following: H. Lob and H. W. Richmond, Proc. London Math. Soc. (2) **31**, 355–369 (1930); H. Lebesgue, Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt [Festschrift Rudolf Fueter) 5–19 (1940); these Rev. **3**, 85; H. Lob, Proc. Cambridge Philos. Soc. **36**, 401–413 (1940); these Rev. **2**, 151. Proofs of Morley's theorem may be found in: MacNeish, Amer. Math. Monthly **31**, 310 (1924); Grossman, ibid. **50**, 552 (1943); these Rev. **5**, 73; Child, Math. Gaz. **11**, 171 (1922); Walls, Edinburgh Math. Notes no. **34**, 12–13 (1944); these Rev. **6**, 100; Lorenz, Deutsche Math. **2**, 587–590 (1937); Hofmann, ibid. **4**, 589–590 (1939); these Rev. **1**, 79. *N. A. Court* (Norman, Okla.).

Cebrian, F. On Hagge's method for the rectification of the circumference with the compass alone. Euclides, Madrid **9**, 223–225 (1949). (Spanish)

The author compares Mascheroni's as well as Hagge's [Revista Soc. Mat. Española **3**, 284 (1914); Wiskundig Tijdschrift **10**, 151 (1913)] method of rectification of the circumference: the former is simpler, the latter more precise. *V. Hlavatý* (Bloomington, Ind.).

Müller, Heinrich. Eine einfache Näherungskonstruktion für die Zahl π. Z. Angew. Math. Mech. **29**, 254 (1949).

Pizá, Pedro A. Triangles cubiques. Mathesis **58**, 10–17 (1949).

Droussent, L. A propos du théorème de Feuerbach. Mathesis **58**, 164–171 (1949).

Marmion, A. Sur la biquadratique de Schröter d'un tétraèdre et les centres des huit sphères tangentes aux faces. Mathesis **58**, 30–43 (1949).

Yang, Chung-Tao. Fixed subplanes of a projective collineation in a finite projective plane. Acad. Sinica Science Record **2**, 262–269 (1949).

The author continues his investigations of collineations in projective planes with coordinates from a Galois field of order p^n [same vol., 157–164 (1948); these Rev. **10**, 58]. He finds the number of fixed subplanes for each type of projective collineation. *M. Hall* (Columbus, Ohio).

Schreiber, Shmuel. On some questions of closure. Riveon Lematematika **3**, 9–13, 54–55 (1949). (Hebrew. English summary)

Let a_0, b_0, c_0, d_0 be 4 points in the plane and consider the transformation associating with them the points a_1, b_1, c_1, d_1 where a_1 is the inverse of a_0 with respect to the circle passing through b_0, c_0, d_0, etc. This note contains some elementary

comments on the iteration and inversion of the above and similar transformations.   *A. Dvoretzky* (Princeton, N. J.).

Fernandez Biarge, Julio. Staudtian cyclic finite projectivities. Revista Mat. Hisp.-Amer. (4) 8, 226–238 (1948). (Spanish)

L'auteur étudie le problème de la détermination des projectivités Staudtiennes cycliques d'ordre fini donné, effectuées sur des figures de première catégorie de même base et avec un corps fondamental de caractéristique zéro. Considérant à cet effet la projectivité Staudtienne la plus générale de l'espèce indiquée, il montre qu'on peut la mettre sous l'une ou l'autre de deux formes normales déterminées. Il s'agit alors de rechercher les corps fondamentaux, les automorphismes et les valeurs des paramètres qui interviennent dans les expressions des projectivités normales obtenus, pour lesquelles ces projectivités sont cycliques d'ordre donné. L'auteur ramène ce dernier problème à celui de la détermination des sous-variétés invariantes dans une transformation en elle-même d'une variété convenablement construite. Ce nouveau problème est souvent facile à résoudre, et l'auteur montre comment il convient d'utiliser les résultats auxquels conduit sa solution en vue de la résolution du problème initial.   *P. Vincensini* (Besançon).

Bilimović, Anton. A natural property of the differential equation of a conic section. Glas Srpske Akad. Nauka 189, 107–118 (1946). (Serbian. Russian summary)

The author discusses some classical metric, intrinsic and projective relations for a conic section. By eliminating the 5 coefficients from the equation of a conic one obtains $9y''y^v - 45y''y'''y^{iv} + 40y'''' = 0$ as the equation that any conic must satisfy. By considering the usual projective definition of a point conic, taking (x, y) and $(x+dx, y+dy)$ as vertices and $((1+d)^\alpha x, (1+d)^\alpha y)$, $\alpha = 1, 2, 3, 4$, as four nearby points, the equality of the cross-ratios of the two pencils (retaining differentials of the 11th order), he obtains the same equation. He also obtains the intrinsic equation of the conic from the above differential equation.
  M. S. Knebelman (Pullman, Wash.).

Deaux, R. Sur deux formes homographiques ternaires. Mathesis 58, 5–10 (1949).

Deaux, R. Décompositions d'une homographie plane non homologique en produit de trois homologies harmoniques. Mathesis 58, 58–68 (1949).

Deaux, R. Décompositions d'une homologie plane en produit de trois homologies harmoniques. Mathesis 58, 151–154 (1949).

Goormaghtigh, R. Sur les triangles de Poncelet. Mathesis 58, 24–29 (1949).

Goormaghtigh, R. Sur le quadrilatère inscriptible. Mathesis 58, 49–52 (1949).

Goormaghtigh, R. Sur l'inversion triangulaire. Mathesis 58, 74–77 (1949).

Goormaghtigh, R. Généralisation du théorème de Pollock. Mathesis 58, 171–173 (1949).

Droussent, Lucien. Sur la conique inscrite dont les foyers sont situés sur l'hyperbole de Kiepert. Mathesis 58, 69–70 (1949).

Gavrilović, Bogdan. Über die Abbildung der Punktmengen in einer transfiniten Menge congruenter projektiver Punktreihen. Glas. Srpske Akad. Nauka 191, 125–138 (1948). (Serbian. German summary)

✳**Salkowski, E. Grundzüge der darstellenden Geometrie.** 4th ed. Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 3. Akademische Verlagsgesellschaft, Leipzig, 1949. xii+194 pp.

This is a reprint of the third edition of Salkowski's textbook which in turn was essentially identical with the second [1943] edition. The second edition was already adapted to a revision of the curriculum of the German Institutes of Technology, which reduced the amount of time available for theoretical subjects. Accordingly the book is less comprehensive than the older texts on descriptive geometry. It aims mainly at stimulating the power of visualizing spatial objects but does not attempt to give the reader a background in the elements of projective geometry. The following list of chapter headings will indicate the scope of the book. I, Orthographic projection on a single plane [27 pp.]. II, Orthographic projection on two planes [horizontal and vertical projections, 32 pp.]. III, Oblique projection and construction of shades and shadows [8 pp.]. IV, The circle and the sphere [32 pp.]. V, Conic sections [13 pp.]. VI, Curved lines and surfaces [18 pp.]. VII, Central (perspective) projection [55 pp.].   *E. Lukacs.*

Krames, J. Über die Flächen konstanter Bildparallaxe und die zugehörigen gefährlichen Raumgebiete. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 8–14 (1948).

Some earlier results of the author on dangerous regions [Monatsh. Math. 52, 265–285 (1948); these Rev. 10, 320] are summarized and supplemented by a brief discussion of the dimensions of the dangerous region.   *E. Lukacs.*

Krames, Josef. Über besondere lineare Büschel von Flächen konstanter Bildparallaxe. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 25–31 (1948).

Continuing his studies of dangerous regions [see the preceding review] the author discusses special families of surfaces with constant y-parallax.   *E. Lukacs.*

Krames, Josef. Allgemeine lineare Büschel von Flächen konstanter Bildparallaxe. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 39–48 (1948).

The author extends statement 5 of an earlier paper [Monatsh. Math. Phys. 49, 327–354 (1941); these Rev. 3, 300] to the case of constant y-parallax. A slightly different generalization of the same statement was given in a recent paper [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa, 156, 233–246 (1948); these Rev. 10, 205]. The y-parallax of two lines is the projection of their parallax on an image plane parallel to the plane formed by the y-direction and the line joining the two centers of projection. For the definition of parallax see the review of the author's earlier paper [these Rev. 10, 205].   *E. Lukacs* (China Lake, Calif.).

Krames, Josef. Zur Fehlertheorie der gegenseitigen Orientierung zweier Luftaufnahmen. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 84, 53–57 (1947).

This is a preliminary discussion of some results of the author's paper [Monatsh. Math. 52, 265–285 (1948); these Rev. 10, 320].   *E. Lukacs* (China Lake, Calif.).

Krames, Josef. Über Bedingungsgleichungen für die Orientierungsunbekannten beim gegenseitigen Einpassen von Luftaufnahmen. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 72–74 (1948).

This is a preliminary announcement of a result giving a relation between the quantities determining the orientation of consecutive frames obtained in aerial surveying.

E. Lukacs (China Lake, Calif.).

Convex Domains, Extremal Problems

Bol, Gerrit. Über konvexe Körper mit Ecken und Kanten. Arch. Math. 1, 427–431 (1949).

By the method of inner parallel bodies the author proves the inequalities

$$M^{*2} \geq \bar{O}O, \quad O^2 \geq 3M^*V, \quad O^3 \geq 27\bar{V}V^2,$$

which in general improve the Minkowski inequalities

$$M^2 \geq 4\pi O, \quad O^2 \geq 3MV, \quad O^3 \geq 36\pi V^2$$

for a convex body B. Here $\bar{O}$ and $\bar{V}$ denote the surface area and the volume of the tangential body $\bar{B}$ of the unit sphere whose planes of support are parallel to the extreme planes of support of B, and $2M^* = \{dO(-\lambda)/d\lambda\}_{\lambda \to +0}$, where $O(-\lambda)$ is the surface area of the inner parallel body of B at distance λ. A less general inequality of the form $M^{*2} \geq \bar{O}O$ has also been proved by Dinghas [Math. Z. 51, 306–316 (1948); Math. Ann. 120, 533–538 (1949); these Rev. 10, 471, 564]. [The reviewer remarks that all these inequalities are special cases of Minkowski's inequalities between mixed volumes.]

W. Fenchel (Los Angeles, Calif.).

Bieri, H. Mitteilung zum Problem eines konvexen Extremalkörpers. Arch. Math. 1, 462–463 (1949).

It has been conjectured that the symmetrical spherical zone which has the minimal M among all convex bodies of revolution with prescribed V and F [Hadwiger, Glur, and the author, Experientia 4, 304–305 (1948); Hadwiger, Portugaliae Math. 7, 73–85 (1948); these Rev. 10, 141, 471] also has this property if all convex bodies are considered. This is, however, not the case. The author gives numerical values of M, V, F for a body (obtained from a sphere by cutting off three suitable segments) which show that this body is "better" than the symmetrical spherical zone.

W. Fenchel (Los Angeles, Calif.).

Hadwiger, Hugo. Elementare Studie über konvexe Rotationskörper. Math. Nachr. 2, 114–123 (1949).

Let M be the integral of the mean curvature, F the surface area, V the volume and a the maximal radius of a parallel circle of a convex body of revolution. The author proves in an elementary way 8 inequalities giving upper and (known) lower bounds for the deficits in Minkowski's quadratic and cubic inequalities between M, F, V. A typical example is $(M - 4\pi a)^2 \leq M^2 - 4\pi F \leq (M - 2\pi a)^2 + 4\pi^2(\pi - 3)a^2$.

W. Fenchel (Los Angeles, Calif.).

Haupt, Otto. Zur Verallgemeinerung des Vierscheitelsatzes und seiner Umkehrung. Ann. Mat. Pura Appl. (4) 27, 293–320 (1948).

Three familiar results from the geometry of plane arcs and curves are the following. (a) Every oval of class C'' with cyclic order at least $2n$ has at least $2n$ vertices. (b) Every convex arc of class C'' with cyclic order 4 has at most 4 vertices. (c) Every oval of class C'' with cyclic order 4 has exactly 4 vertices. In this paper the author has obtained generalizations of these and other related theorems in a completely topological fashion, thus revealing the essentially topological, rather than metric, framework on which the theorems rest.

The family of circles which are fundamental in the above theorems are replaced by a k-parameter family of arcs or curves called order characteristics which have certain postulated properties. An arc or curve is said to have k-order r if r is the maximum number of points, other than fixed points, in which it is met by any order characteristic. A point P of an arc B is said to have k-order r if every sufficiently small neighborhood of P on B has this k-order; P is called a k-vertex if it has k-order at least $k+1$. The following are among the results established. (i) If an arc B (or curve C) has at least k-order $kt+1$ with k, $t \geq 1$, it has at least t (or $t+1$) k-vertices. (ii) Every arc B containing only isolated k-vertices is the sum of closed arcs, disjoint except for endpoints, each of k-order k and such that the k-vertices interior to B are endpoints of these arcs. (iii) If B (or C) is an arc (or curve) having at least k-order $p \geq k+1$, the number of k-vertices on B (or C) is at least $p-k$ (or p). (iv) If the arc B has at most k-order $k+1$, the number of k-vertices is at most $k+1$. (v) The curve C has exactly k-order $k+1$ if and only if C has exactly $k+1$ k-vertices. (vi) Every isolated k-vertex has exactly k-order $k+1$. Every arc with exactly one vertex has k-order $k+1$.

The proofs depend heavily on properties of order characteristics developed in earlier papers by the author. It is noteworthy that for results (i) and (ii) it is not necessary to make any assumption at all corresponding to the existence of osculating circles in the usual case, and in no case is it essential to assume more than the existence of unique limiting arcs of order characteristics at the vertices. The theorems in this paper contain as special cases a substantial body of earlier results concerning vertices or sextactic points on ovals and convex arcs. They do not, however, appear to include such results as the extension of the four-vertex theorem to all simple closed curves of class C'', since the order characteristics are required to meet an arc or curve in points having the same order on both.

S. B. Jackson (College Park, Md.).

Haupt, Otto. Über eine Kennzeichnung der Ovale von der zyklischen Ordnung Vier. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1947, 73–74 (1949).

A brief discussion of topological generalizations of the four-vertex theorem and two other related theorems. These are discussed in more detail in another paper by the same author; see the preceding review.

S. B. Jackson.

Inzinger, Rudolf. Ein Beweis des Vierscheitelsatzes für Eilinien. Anz. Akad. Wiss. Wien. Math.-Nat. Kl. 83, 13–14 (1946).

A proof of the four-vertex theorem based on a result of the author concerning the extreme values of the tangential distance from a given fixed circle to the osculating circles of an oval.

S. B. Jackson (College Park, Md.).

Marchaud, A. Sur les ovales. Ann. Soc. Polon. Math. 21 (1948), 324–331 (1949).

Let us call the locus of the points conjugate to a point P with respect to an oval O (in the sense of the theory of the conics) the polar of P. The author proves (together with some analogous propositions concerning this range of prob-

lems) that if there exists a set of points having a point of accumulation lying in the interior of the region bounded by O such that any point has a rectilinear polar, then O is an ellipse. It is shown that if the point of accumulation lies on O the above statement does not hold. *L. Fejes Tóth.*

Borsuk, Karol. Correction à mon travail "Sur la courbure totale des courbes fermées." Ann. Soc. Polon. Math. **21** (1948), 302 (1949).

In the paper in question [same Ann. **20** (1947), 251–265 (1948); these Rev. **10**, 60] the author applies the usual integral formula for the arc length of a curve. But the assumptions made on the parametric representation of the curve do not secure the validity of this formula. Here sufficiently strong assumptions are given. The only influence of the error is that the theorem is proved for a slightly more restricted class of curves than stated in the paper. *W. Fenchel* (Los Angeles, Calif.).

Algebraic Geometry

Du Val, Patrick. On absolute and non absolute singularities of algebraic surfaces. Rev. Fac. Sci. Univ. Istanbul (A) **11** (1944), 159–215 (1946). (English. Turkish summary)

A singular point of an algebraic surface is said to be absolute if, however the surface be transformed birationally into a nonsingular surface, the point in question necessarily corresponds to a curve (or set of curves), with or without further isolated points. The author proves the following results. (I) Every absolute singularity is proper (in the sense of Severi). (II) It is possible for an algebraic surface to have an infinite sequence of singular points, each in the neighbourhood of its predecessor, each of which is proper but not absolute; or each of which is absolute. A large part of the paper is devoted to an outline of the classification of improper singularities, illustrated by a number of examples.

The paper involves a mass of detail not readily condensed in a précis. But it is perhaps worth while quoting the examples given by the author to illustrate the possibilities in (II). These are provided by considering, for $(m, n) = (4, t)$ or $(5, 6)$, respectively, the surfaces in ordinary space with equations $\phi^m = \psi^m$, where $\phi = 0$, $\psi = 0$ are surfaces of order n, m which touch at a point P. A sequence of singular points with the stated property has P as the first point, the next point being consecutive to P along the curve $\phi = \psi = 0$, and the remaining points of the sequence being consecutive to this along the neighbourhood of P. *J. A. Todd.*

Abellanas, Pedro. Decompositions produced by a collineation in $P_k{}^n$. Revista Mat. Hisp.-Amer. (4) **8**, 261–276 (1948). (Spanish)

The author discusses the decomposition of a projective space $P_k{}^n$ (over an arbitrary field k) into the sum of self-corresponding spaces for a linear transformation φ. Cohn-Vossen [Math. Ann. **115**, 80–86 (1937)] has discussed this question in the case where k is the complex number field. Some reformulation of the problem is required in case k is not algebraically closed. The principal result of this paper is that the necessary and sufficient condition for the existence of a unique "canonical" decomposition (in a suitably defined sense) is that φ admits no ~~invariant subspace of positive dimension.~~ variety of *D. B. Scott* (London). ambient elements. By this, the author intends an odd-dimensional subspace a^{2k+1} of $P_k{}^n$ each of whose points lies in a minimal self-corresponding space properly contained in a^{2k+1}.

Andreotti, Aldo. Sulle corrispondenze fra due curve birazionalmente distinte a moduli generali e sui modelli minimi dei loro prodotti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 375–380 (1948).

Let C and Γ be curves of genus p and π, respectively, with general moduli; and let C be not birationally equivalent to Γ. The author's aim is to find the order of the minimum models of the surface $F = C \times \Gamma$; as a minimum model one has to understand a surface which has the minimum order among all the surfaces that are birationally equivalent to F without exceptions. The author first remarks, by a degeneration method, that only correspondences with zero valence may subsist between C and Γ. Hence it follows that for any curve D of the surface F (since any curve D may be considered as a correspondence between C and Γ) the algebraic equivalence: (1) $D \equiv nA + \nu B$ (n, ν integers) is satisfied; in (1) $A = x \times \Gamma$, $B = y \times C$, and x and y represent points of C and Γ, respectively. If D defines the linear system $|D|$, one may deduce from (1) that $|D|$ has the virtual grade $2n\nu$. Therefore, if D is chosen so that $|D|$ is at least ∞^3, with no basis points and no neutral pairs, and, besides, its grade is minimum, the projective image of F, which is obtained by means of $|D|$, is a minimum model of F, of the order $2n\nu$. Hence, by simple considerations, and using some of Severi's results [Vorlesungen über algebraische Geometrie, Teubner, Leipzig-Berlin, 1921, Anhang G], one may deduce that the minimum models of F have the order $2n\nu$, where $n = 1$, for $p = 0$; $n = p - [p/3] + 2$, for $p = 1$, 3; $n = p - [p/4] + 3$ for $p \neq 0$, 1, 3; similarly, if we exchange p with π, and n with ν. Some remarks follow: the normal minimum models have no singularities; if C or Γ is a rational curve, the minimum models are ruled surfaces. Also the number of the projectively distinct minimum models is determined. *F. Conforto* (Rome).

Roth, Leonard. Some arithmetical questions in the theory of the base. Ann. Mat. Pura Appl. (4) **27**, 115–134 (1948).

Let V_k be an algebraic variety, free from singularities, and let $(A, B, C, \cdots)$ be a minimum base of ρ (>1) manifolds V_{k-1} on V_k. Then any V_{k-1} of V_k can be algebraically expressed in the form $xA + yB + zC + \cdots$, where $x, y, z, \cdots$ are integers or zero, and its virtual grade $[(xA + yB + zC + \cdots)^k]$ is given by the fundamental form $f(x, y, z, \cdots)$, where f is a homogeneous polynomial of degree k in the variables $x, y, z, \cdots$, whose coefficients are integral constants depending on the virtual characters $[A^k]$, $[A^{k-1}B]$, etc., of the base manifolds.

Any birational transformation, without exceptional elements, of V_k in itself induces an integral automorphism on the fundamental form $f(x, y, z, \cdots)$. The converse is not true, since an integral automorphism of f may or may not arise from self-transformations of V_k (the author calls it then an effective or virtual automorphism of f, respectively), and a given automorphism of f may be associated with more than one and even with an infinity of self-transformations of V_k. However, the preceding remark may be utilized in determining the birational self-transformations of V_k which are free from singular elements, or present given exceptions, as it has been shown in the case of surfaces, e.g., for $k = 2$, by F. Severi [Rend. Circ. Mat. Palermo **30**, 265–288 (1910)], L. Godeaux [Bull. Int. Acad. Sci. Cracovie Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1913**, 529–547; Bull. Soc. Roy. Sci. Liége **11**, 331–335 (1942); these Rev. **7**, 73], F. R. Sharpe and V. Snyder [Trans. Amer. Math. Soc. **15**,

266–276 (1914)], G. Fano [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 29_1, 408–415, 485–491 (1920)], T. G. Room [Proc. Roy. Soc. London. Ser. A. 193, 25–43 (1948); these Rev. 9, 610]; other surfaces admitting an infinite discontinuous group of birational self-transformations are studied in the first section of this paper.

For varieties of dimension higher than two, no comparable theory existed as yet. The second and third sections deal respectively with types of V_3 and V_4 with base-number $\rho=2, 3$, and present a large amount of experimental material, which shows many relations between algebraic and arithmetical questions, as, e.g., those suggested by the diverse effect of the exceptional elements. The automorphisms of the fundamental forms are studied by using the results of B. Segre [Proc. Cambridge Philos. Soc. 41, 187–209 (1945); Univ. Nac. Tucumán. Revista A. 5, 7–68 (1946); these Rev. 7, 169; 9, 170]; from some of these results a simple criterion is derived for deciding whether a V_4 with base-number 2 possesses a self-transformation free from exceptional elements and, in particular, whether a given irreducible curve or surface in S_4 can be the base of a symmetrical Cremona transformation in S_4 which has no other base elements. Here is a list of the V_k's investigated in the paper: $k=2$, $\rho=2$: (1) the intersection, residual to a plane, of a planar $V_3{}^3$ of S_5 with a cubic primal; (2) the intersection of a $V_4{}^6$ of C. Segre with a prime and a quadric primal. $k=2$, $\rho=3$: (3) the quartic surface of S with two conic nodes; (4) the intersection of three quadric primals of S_5 having two skew lines in common. $k=3$, $\rho=2$: (5) the intersection of a $V_4{}^6$ of C. Segre with a quadric primal; (6), (7), (8) the V_3 representable in S_3 by means of surfaces (of a sufficiently high order) passing through a sextic of genus three, or a twisted cubic, or an elliptic twisted quartic; (9) the planar $V_3{}^3$ of S_5. $k=3$, $\rho=3$: (10), (11), (12), (13) the V_3 representable in S_3 by means of surfaces passing through a sextic of genus three and an isolated base point, or an elliptic quartic and an isolated base point, or two skew lines, or two elliptic quartics; (14) the product of a curve and a surface; (15) the product of three curves; (16) the quartic primal of S_4 with two conic nodes; (17) the $V_3{}^3$ intersection of three quadric primals of S_6, with two of its lines as base curves; (18) the intersection of a $V_4{}^6$ of C. Segre with a cubic primal passing through two planes of $V_4{}^6$ of the same system. $k=4$, $\rho=2$: (19) a quartic primal of S_5 containing a quartic scroll R, with R as base surface; (20) a linear space S_4, with the intersection of three general quadric primals as base curve. $k=4$, $\rho=3$: (21), (22) a linear space S_4, with a quadric surface and an isolated point, or a Del Pezzo quartic surface and an isolated point, as base elements. *B. Segre* (Bologna).

Sz.-Nagy, Gyula. Darstellung algebraischer Flächen von Gestalt einer Kurve. Hungarica Acta Math. 1, no. 4, 10–11 (1949).

If $f(x, y)=0$ is a real plane curve, the real sheets of the surface $z^2+f^2(x, y)-\epsilon^2=0$, where ϵ is sufficiently small, enclose a thin tubular region enclosing the curve. The surface has a singularity at the point at infinity on the z-axis, at those of the plane curve $f^2(x, y)-\epsilon^2=0$ and nowhere else. If a curve in space is given by equations of the cylinder-and-monoid form $f(x, y)=0$, $z=g(x, y)/h(x, y)$, the surface

$$[z, h(x, y)-g(x, y)]^2+f^2(x, y)-\epsilon^2=0$$

is similarly related to the curve, save that a line parallel to the z-axis and satisfying

$$f^2(x, y)-\epsilon^2=g(x, y)=h(x, y)=0$$

(if these equations have any common solutions) is a locus of isolated singularities. *P. Du Val* (Athens, Ga.).

Fano, Gino. Su una particolare varietà a tre dimensioni a curve-sezioni canoniche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 151–156 (1949).

La variété qui représente les cordes d'une réglée rationnelle normale R^4 de S_8 qui appartiennent à un complexe linéaire K est une $M_3{}^{22}$ de S_{13} à sections curvilignes canoniques. La variété $M_3{}^{22}$ contient deux réglées: l'une d'elles ρ^6 est une réglée rationnelle normale de S_7. Elle représente les faisceaux de K dans les plans des coniques de R^4, dont les sommets décrivent une cubique gauche γ. L'autre ξ^{22} représente les faisceaux de K ayant leur sommet sur R^4 et dont le plan passe par une génératrice de R^4: la courbe lieu des sommets est une C^{12} de genre virtuel 9 ayant 4 points doubles sur γ. La réglée ξ^{22} appartient au système linéaire double du résiduel $|F^{16}|$ de ρ^6 par rapport aux sections hyperplanes. La variété $M_3{}^{22}$ contient un système linéaire $|F^{14}|$ de dimension 5 et de degré 4, de sorte que $M_3{}^{22}$ est birationnellement équivalente à l'intersection de deux hyperquadriques de S_5: elle est donc rationnelle. Elle est la projection d'une $M_3{}^{22}$ de S_{13} à sections curvilignes canoniques, à partir d'un S_4 qui la coupe suivant une quartique rationnelle normale. L'auteur termine par l'examen de deux cas particuliers: celui où K contient les génératrices de R^4 et celui où la cubique γ est tracée sur R^4. *L. Gauthier.*

Galafassi, Vittorio Emanuele. Questioni di realità sulle curve trigonali reali. Ann. Mat. Pura Appl. (4) 27, 135–151 (1948).

Le théorème de Harnack [Math. Ann. 10, 189–198 (1876)] assure l'existence de courbes algébriques réelles de genre p à modules généraux, ayant k branches ($0 \leq k \leq p+1$). Ce théorème est encore valable pour les courbes hyperelliptiques [Klein, Math. Ann. 42, 1–29 (1893)]. L'auteur étudie le cas des courbes trigonales (contenant une $g_3{}^1$) au moyen de leur représentation par une courbe canonique, en examinant la réalité des génératrices et directrices de la réglée rationnelle normale R des trisécantes. L'espèce m d'une courbe trigonale est l'ordre de la directrice minima de R et on a $(p-4)/3 \leq m \leq (p-2)/2$. L'auteur démontre que le nombre k des branches réelles d'une courbe trigonale réelle de genre p et d'espèce m vérifie: $\epsilon \leq k \leq p+1$ où $\epsilon=0$ pour $p=3$, pour p et m simultanément pairs, et pour $m=(p-2)/2$; enfin $\epsilon=1$ dans les autres cas.

L'auteur montre qu'il existe effectivement des courbes correspondant à toutes les valeurs de k dans cet intervalle, en s'appuyant sur une étude qu'il a faite antérieurement des courbes réelles tracées sur des réglées rationnelles [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 827–831, 922–927 (1946); ces Rev. 8, 402]. D'autre part, si ν désigne le nombre de points réels du jacobien de $g_3{}^1$, il existe des courbes pour lesquelles ν prend toutes les valeurs paires telles que $k-1 \leq \nu/2 \leq p+2$, pour lesquelles il existe des groupes réels de $g_3{}^1$ ayant un seul point réel. Pour $k=1, 2, 3$ il peut arriver que tout groupe réel G de $g_3{}^1$ ait trois points réels: pour $k=3$ il y a un point sur chaque branche; pour $k=2$ il y a un point sur une branche impaire et deux points sur une branche paire. Des exemples explicites sont donnés pour le genre 3, qui sert de départ à une récurrence sur p. *L. Gauthier* (Nancy).

Tanturri, Giuseppe. Inviluppi di piani che secano proiettivamente una F^3 di S_3.   Boll. Un. Mat. Ital. (3) **4**, 48–52 (1949).

Les surfaces du troisième ordre de S_3 à sections planes toutes projectivement identiques sont les réglées, rationnelles ou coniques [Fano, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **1**, 473–477 (1925)]. L'auteur montre, en s'appuyant sur un mémoire antérieur [même Boll. (3) **3**, 46–48 (1948); ces Rev. **10**, 207] que l'enveloppe des plans qui coupent une F^3 de S_3 suivant des cubiques de module donné générique est de classe douze. Quand la cubique section est harmonique ou équianharmonique, la classe s'abaisse respectivement à six et quatre. L'enveloppe n'est décomposée, pour une valeur générique du module, que dans deux cas. Lorsque F^3 a un point double uniplanaire U, l'enveloppe comporte le point U avec une certaine multiplicité, et la partie résiduelle est de même classe que F^3. Lorsque F^3 a trois points doubles biplanaires, on a un faisceau tangentiel de telles surfaces du troisième ordre et de troisième classe ayant leurs singularités fixes. Chacune est l'enveloppe des plans qui coupent les autres suivant des cubiques de module constant. Une des enveloppes du faisceau est constituée des trois points doubles. Une autre est formée d'un point triple tangentiel: le point de concours Ω des plans osculateurs à F^3 le long des droites qui joignent les points doubles deux à deux. Tous les plans passant par Ω coupent F^3 suivant des cubiques équianharmoniques.

L. Gauthier (Nancy).

Differential Geometry

✱**Bol, Gerrit.** Differentialgeometrie.   Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 163–185. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Among the papers described briefly in this set of reviews are the following unpublished manuscripts. In general surface theory, a manuscript by M. Pinl on examples and a classification of surfaces in R_4 with isotropic mean curvature vectors, and another manuscript on complex minimal surfaces in R_5 with vanishing Gaussian curvature. In the projective differential geometry section, there are reviews of manuscripts by W. Süss, "Grundlegung einer projektiven Flächentheorie unter Benutzung von Eichflächen," and by K. Strubecker [see abstracts in Anz. Akad. Wien. Math.-Nat. Kl. **78**, 90–94 (1941); Ber. Math.-Tagung Tübingen 1946, pp. 136–139 (1947); these Rev. **8**, 487; **9**, 62], and there is also a description of two series of manuscripts by G. Bol on some new results in the differential geometry of curves in a three-dimensional projective space, and on a projective surface theory based on a new differentiation process.   *A. Schwartz* (New York, N. Y.).

Pinl, M. Binäre orthogonale Matrizen und integrallose Darstellungen isotroper Kurven.   Math. Ann. **121**, 1–20 (1949).

The algebraic foundations of the differential geometry of isotropic curves consists essentially of statements concerning the rank of certain Gram matrices G whose elements involve the derivatives of the position vector of a curve up to a certain order. These statements concerning rank are therefore differential equations for the n unknown functions $x_1(t), \cdots, x_n(t)$ of the complex variable t, i.e., the n components of the position vector. Since the equations are nonlinear they belong to the class of special Monge problems whose solution can be obtained from the complete integral of the associated Hamilton equation by differentiation and elimination only. In the dualistic theory of isotropic and related curves there occur Gram matrices H whose elements relate to the dual Hamilton integral curves. Statements of rank concerning G can then be replaced by statements of rank concerning H. In many cases the differential equations originating from the matrices G can be replaced by purely algebraic equations of condition upon the matrix H. In this paper it is shown how this can be done by the use of binary orthogonal matrices.   *E. T. Davies.*

Löbell, Frank. Ein vektorielles Seitenstück zum Gauss-Bonnetschen Integralsatz.   S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 119–128 (1949).

For a simple closed curve γ which bounds an area A on a regular surface the Gauss-Bonnet theorem states that:

$$\int_\gamma G ds + \sum \alpha = 2\pi - \int_A K df,$$

where G is the geodesic curvature of γ, α are its outer angles, and K is the Gaussian curvature of the surface. The author develops a vectorial supplement to this theorem, valid for surfaces imbedded in Euclidean space:

$$\int_\gamma \mathbf{g} ds = 2 \int_A K df,$$

where $\mathbf{g} = \mathbf{n} \times d\mathbf{n}/ds$, $\mathbf{n}$ being the unit normal to the surface, and df the vectorial area element of A. This result is applied to find an analogue of the Gauss-Bonnet theorem for surface-strips in the form

$$\int_\gamma G ds + \sum \alpha = F - 2n\pi,$$

where γ is the mid-line of the surface strip, and F is the area on the unit sphere swept out by the "quarter-arcs" (of length $\pi/2$) tangent to the spherical image of the normals to the strip along γ. When the strip is tangent to an actual surface, this reduces to the Gauss-Bonnet theorem.

C. B. Allendoerfer (Cambridge, Mass.).

Löbell, Frank. Betrachtungen über Flächenabbildungen. II. Rissmassstab und Querrissmassstab.   S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 15–23 (1949).

[For part I cf. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 175–183 (1947); these Rev. **9**, 614.] A mapping $x(u, v) \rightarrow y(u, v)$ of two surfaces in Euclidean 3-space determines a correspondence between vectors dx and dy in the tangent planes at corresponding points. The author introduces two concepts: the "projection measure" n (Rissmassstab) and the "transverse measure" q (Querrissmassstab) of the mapping at any given point. They are defined respectively by the equations $n = (dx \cdot dy)/dx^2$ and $q = (x_u \times x_v) \cdot (dx \times dy)/|x_u \times x_v| dx^2$. Directions for which n takes on extreme values are called principal directions, and formulas are developed for n and q closely analogous to the usual ones for normal curvature and geodesic torsion in surface theory. Geometric interpretations of certain invariants of surface mappings can be given in terms of q and n. When the surface $y(u, v)$ is the usual spherical image of $x(u, v)$, n and q are precisely the normal curvature and geodesic torsion of x. The usual theory of curvature of a

surface is thus viewed as a part of the theory of maps of pairs of surfaces. *S. B. Jackson* (College Park, Md.).

Löbell, Frank. Betrachtungen über Flächenabbildungen. III. "Gleichmässige" Abbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 25–33 (1949).

A mapping $x(u, v) \rightarrow y(u, v)$ of two surfaces determines a transformation of their tangent planes at any given pair of corresponding points. The transformation is called uniform (gleichmässig) at a point provided the "projection measure" [see the preceding review] at this point is independent of the direction. The author here discusses uniform mappings both with respect to their geometric interpretation and their characterization in terms of differential invariants of the mapping. *S. B. Jackson* (College Park, Md.).

Löbell, Frank. Betrachtungen über Flächenabbildungen. IV. Ausgezeichnete Abbildungen verschiedener Art. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 35–43 (1949).

Let $x(u, v) \rightarrow y(u, v)$ be a mapping of two surfaces in Euclidean space. Three special types of such mappings are characterized respectively by the following three properties: (a) the "projection measure" n [see the second preceding review] is independent of the direction; (b) the scalar differential invariant $J(u, v) = (x_u \cdot y_v - x_v \cdot y_u)/|x_u \times y_v|$ vanishes; (c) the vector differential invariant $\mathfrak{J}(u, v) = (x_u \times y_v - x_v \times y_u)/|x_u \times x_v|$ vanishes. [Cf. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 217–237 (1944); these Rev. **8**, 89.] This paper contains several results on these types of mappings. A typical theorem is the following. If the mapping $x(u, v) \rightarrow y(u, v)$ and the mapping $x(u, v) \rightarrow z(u, v)$ are both of type (c) above, the same is true of all the mappings $x(u, v) \rightarrow \alpha y(u, v) + \beta z(u, v)$, where α and β are arbitrary constants. *S. B. Jackson*.

Löbell, Frank. Betrachtungen über Flächenabbildungen. V. Flächenpaare mit vorgegebener Schiefe. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 77–80 (1949).

[Cf. the three preceding reviews.] The obliqueness [Schiefe] of the mapping $x(u, v) \rightarrow y(u, v)$ of two surfaces in space is a scalar function defined by the equation $J(u, v) = (x_u \cdot y_v - x_v \cdot y_u)/|x_u \times x_v|$. The author here determines, for a given surface $x(u, v)$, all those surfaces $y(u, v)$ for which the obliqueness of the map is given by a prescribed function $f(u, v)$. The solution contains two arbitrary functions. *S. B. Jackson* (College Park, Md.).

Löbell, Frank. Betrachtungen über Flächenabbildungen. VI. Gerade Abbildungen mit Nebenbedingungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 179–186 (1949).

This is a completion of paper V [see the preceding review] concerning the determination of surfaces $y(u, v)$ on which a given surface $x(u, v)$ is mapped with an obliqueness of zero, i.e., $J(u, v) = 0$. The development is carried out with the use of the differential operator $D = (1/|x_u \times x_v|)(x_v \partial/\partial u - x_u \partial/\partial v)$.
S. B. Jackson (College Park, Md.).

Lalan, Victor. Le rôle des asymptotiques virtuelles dans la théorie de l'immersion des surfaces. Bull. Sci. Math. (2) **73**, 16–32 (1949).

The problem of imbedding a surface in Euclidean space depends upon the solution of the Gauss and Codazzi equations. By a process of elimination it may be made to depend upon the solution of two partial differential equations of the third order in a single unknown. This elimination goes particularly smoothly because of the linearity of the Codazzi

equations. The author first discusses the general theory of such eliminations and then applies it to the Codazzi equations. The most convenient process of elimination utilizes as unknowns the direction numbers of the asymptotic lines. The author points out similar conclusions in the work of Darboux. The paper concludes with a discussion of the relation of the above results to the fact that the indicatrix of an asymptotic net is a function of the asymptotic directions. *C. B. Allendoerfer* (Cambridge, Mass.).

Saban, Giacomo. Due problemi variazionali nella teoria metrica delle superficie rigate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 207–213 (1949).

Le premier des deux problèmes indiqués dans le titre consiste dans l'étude de la variation première de l'arc sphérique dualistique S d'une surface réglée. La surface réglée R étant définie par le vecteur dualistique $G_1(s)$ ($s = $ arc de l'indicatrice sphérique des génératrices) et R^*, ayant en commun avec R les deux génératrices relatives aux valeurs 0 et S de l'arc dualistique, étant définie par $G_1^* = G_1 + UG_2 + VG_3$ ($U = \eta \bar{U}$, $V = \eta \bar{V}$), le développement de l'arc sphérique dualistique S^* de R^* conduit à la formule $\delta S = - \int_0^S \delta \sigma d V dS$ où σ est la courbure sphérique dualistique de R. De cette formule l'auteur déduit que si l'on considère deux rayons d'une congruence rectiligne et les différentes surfaces réglées de la congruence contenant ces rayons, les extremums de la longueur d'arc d'indicatrice des génératrices et du paramètre distributeur intégral entre les deux rayons sont réalisés par les conoïdes droits. Si R et R^* sont à plan directeur, se correspondent avec parallélisme des génératrices rectilignes homologues et ont en commun un couple de génératrices distinctes, le paramètre distributeur intégral est le même pour les deux portions de surfaces comprises entre les deux génératrices communes; il résulte de ce théorème que le paramètre distributeur est nul pour la portion continue d'une surface réglée comprise entre deux génératrices concourantes, si celles-ci existent. Pour la variation première de la courbure sphérique dualistique σ lorsqu'on passe de R à une surface réglée infiniment voisine R^* l'auteur obtient la formule $\delta \sigma = \sigma^2 V - \sigma U' + (d/ds)(V' + \sigma U) + V$, puis, généralisant un problème de Radon, il considère l'intégrale $J = \int \Phi(\sigma) dS$, et trouve que les conditions d'extrémum de J sont: $(1 + \sigma^2) \cdot \Phi' + d^2 \Phi'/dS^2 - \sigma \Phi = 0$, $\Phi' \cdot \sigma - d\Phi/dS = 0$, et montre que les surfaces extrémales du problème variationnel pour J s'obtiennent toutes avec un nombre fini de quadratures. Dans le cas particulier où $\Phi(S)$ est une constante dualistique, les extrémales sont les conoïdes de la congruence des droites qui coupent normalement la perpendiculaire commune aux deux génératrices entre lesquelles on intègre.
P. Vincensini (Marseille).

Müller, Hans Robert. Die Böschungslinien des elliptischen Raumes. Monatsh. Math. **53**, 151–164 (1949).

The author applies a Blaschke-Cartan method expounded and used in previous papers [Monatsh. Math. **52**, 138–161, 181–188 (1948); these Rev. **10**, 145, 326] to the study of cylindrical helices in a three-dimensional elliptic space (E_3), the curvature of which is taken to be 1. By definition a cylindrical helix in (E_3) is a curve (H): $P_0 = P_0(v)$ lying upon a Clifford left (or right) cylinder (C) and cutting the generators G of (C) under a constant angle θ. Notations: (P_0, P_1, P_2, P_3), (r_1, r_2, r_3) and (r_1', r_2', r_3') denote the canonical repère (autopolar tetrahedron) and the associated Euclidean trihedrals attached to (H) at the point P_0, $\{P_0 P_1\}$ the tangent, $\{P_0 P_2\}$ the principal normal, $\{P_0 P_3\}$ the binormal, Q_0 the conjugate point of P_0 on the generator $G(P_0)$

of (C) through P_0, q_0 and q_0' the left and right direction vectors of $G(P_0)$. From the simple formulae

$$Q_0 = P_1 \cos \theta + P_2 \sin \theta, \quad q_0 = r_1 \cos \theta + r_2 \sin \theta,$$
$$q_0' = r_1' \cos \theta + r_2' \sin \theta$$

are derived the following results. The principal normal of (H) at P_0 coincides with the normal to (C) at P_0. The curvature κ_0 and the torsion τ_0 of (H) are related by $\tau_0 + 1 = \kappa_0 \cot \theta$. The curve $P_3 = P_3(v)$ is a cylindrical helix (H_3); its curvature κ_3 and its torsion τ_3 are linked with κ_0 and τ_0 by $\tau_3 \tau_3 = 1$, $\kappa_0^{-1} + \kappa_3^{-1} = \cot \theta$. The curves (H) and (H_3) are lines of striction of the ruled surface generated by their common binormal $\{P_0, P_3\}$. The author investigates (H) in connection with the orthogonal trajectories on (C) of the generators. Another cylinder (C^*) is placed through (H) such that (H) is the intersection of (C) with (C^*). Relations between the spherical images (r) and (r') of (H) and the spherical images of the generators of (C) and (C^*) are established. The paper ends with two properties of Euclidean cylindrical helices, the transpositions of which to the elliptic case are not valid.   *C. Y. Pauc* (Cape Town).

Hsiung, Chuan-Chih. Invariants of intersection of certain pairs of space curves. Bull. Amer. Math. Soc. 55, 623–628 (1949).

(I) Soient deux courbes $C, \bar{C}$ se coupant en un point ordinaire O avec des tangentes distinctes $t, \bar{t}$ et des plans osculateurs distincts dont la droite commune est désignée par p. L'auteur détermine à partir des éléments différentiels du troisième ordre au plus des courbes C et $\bar{C}$ au point O, un invariant projectif désigné par I lorsque la droite p est distincte de t et $\bar{t}$, par J lorsque la droite p coïncide avec la droite $\bar{t}$. Si $R, \bar{R}$ et $T, \bar{T}$ sont respectivement les rayons de courbure et de torsion des courbes $C, \bar{C}$ en O, on a $I = -R\bar{T}/\bar{R}T$, $J = (\frac{2}{3})^3 (R^2 \bar{R}^4)/(T^3 \bar{T})$. Puis, ces invariants sont interprétés projectivement comme birapports de quatre cônes simples passant par t et $\bar{t}$ dans le premier cas, de quatre droites dans le second cas.

(II) Soient enfin, deux courbes $C, \bar{C}$ se coupant en un point ordinaire O, avec des tangentes distinctes $t, \bar{t}$ mais avec le même plan osculateur. L'auteur trouve un invariant projectif déterminé par les éléments différentiels du troisième ordre au plus, et qui est $K = -\bar{T}/T$. L'auteur interprète K comme birapport de quatre droites simples du plan osculateur commun.   *M. Decuyper* (Lille).

Hsiung, Chuan-Chih. Invariants of intersection of certain pairs of curves in N-dimensional space. Amer. J. Math. 71, 678–686 (1949).

It is well known that there is a projective tac-invariant of Mehmke-Smith associated with two plane curves having ordinary contact at a nonsingular point. A number of authors have extended this result to two curves C, C' in n-dimensional space having the same tangent at an ordinary point O, by imposing different conditions on their respective osculating linear spaces at the point O. In particular, B. Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 22, 392–399 (1935)] considered the most general case in which the two curves C, C' have at the point O the same osculating linear spaces of dimensions $1, \cdots, r$, where r is any fixed integer satisfying $1 \leq r \leq n-1$. In this paper the author studies the situation in which the two curves C, C' have distinct tangents at the common point O. This investigation generalizes certain results for ordinary space obtained by the author in his paper reviewed above.

The first two sections are concerned with the case in which the two curves have distinct osculating linear spaces at O. Let two curves C, C' in n-dimensional projective space S_n ($n \geq 3$) intersect at an ordinary point O with osculating linear spaces S_k, S_k' ($k = 1, \cdots, n-1$), respectively. Let $x_1, \cdots, x_n$ represent projective nonhomogeneous coordinates of a point in the space S_n. Let the point O be the origin and let the x_i-axis be the line of intersection of the osculating linear spaces S_i and S_{n-i+1}; then the power series expansions of the two curves in the neighborhood of the point O may be written in the form $C : x_i = a_i x_1^i + \cdots$ ($i = 2, \cdots, n$), $C' : x_i = b_i x_n^{n-i+1} + \cdots$ ($i = 1, \cdots, n-1$). A set of projective invariants I_i ($i = 2, \cdots, n-1$) of the two curves C, C' at the point O is found to be defined by the expressions $I_i = b_i^{n-i}/a_i^{i-1}(a_i/b_i)^{n-1}$. The invariants I_i are characterized metrically by an algebraic expression of the ith curvatures ρ_i, σ_i of C, C' at O, respectively, where $i = 1, 2, \cdots, n-1$. A projective characterization of I_i is obtained as follows. Let the "point at infinity" on the x_i-axis ($i = 1, 2, \cdots, n$) be denoted by O_i and let Γ_i, Γ_i' be the projections of the curves C, C' from the center $O_2 \cdots O_{i-1} O_{i+1} \cdots O_{n-1}$ upon the space $OO_1 O_i O_n$ ($i = 2, \cdots, n-1$). In the space $OO_1 O_i O_n$ consider the system of cones of order $n-1$ and vertex O such that the polar surfaces of orders $1, \cdots, n-i-1, n-i+1, \cdots, n-1$ of any point in the plane $OO_i O_n$ with respect to any cone of the system (the last of these polars being the cone itself) all pass through the line OO_1. Among the cones of this system a unique cone K can be determined which has the following properties. (1) It has contact of order $i(n-1)$ with the curve Γ_i at O. (2) It passes through the tangent of the curve Γ_i' at O. (3) The polar surfaces of orders $1, \cdots, n-2$ of any point in the plane $OO_1 O_n$ with respect to this cone all pass through the line OO_i. Similarly a cone K' can be determined which has contact of order $(n-1)(n-i+1)$ with the curve Γ_i' and which also has properties (2) and (3). The two cones K and K' determine a pencil of cones of order $n-1$ and vertex O in the space $OO_1 O_i O_n$. Belonging to this pencil there are two degenerate cones K_1 and K_2 which consist of the plane $OO_1 O_n$ (counted $n-i$ times), and the plane $OO_1 O_i$ (counted $i-1$ times), respectively. The invariant I_i is equal to the cross-ratio of the four planes K_1, K_2, K, K'. Somewhat similar characterizations of invariants I_i and J_i are obtained for the case in which the two curves C, C' have certain common osculating linear spaces.   *P. O. Bell* (Lawrence, Kan.).

Blaschke, Wilhelm. Zur Bewegungsgeometrie auf der Kugel. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1948, no. 2, 9 pp. (1948).

Using "moving frames," the author develops the formulae for a (closed, i.e., initial state = final state) one-parameter motion of the plane and the 2-sphere, and discusses the concepts of space pole, body pole, Steiner point, and Steiner vector. A relation between the total geodesic curvature of the path of an arbitrary point x, the Steiner vector, and the index of the body pole curve with respect to x is obtained.   *H. Samelson* (Ann Arbor, Mich.).

Laura, Ernesto. Sopra un gruppo di condizioni necessarie affinchè un ds^2 sia di classe h. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 99, 97–113 (1940).

(1) Let b_{es} and R_{abcd} be the second fundamental tensors and the curvature tensor, respectively, of a V_m in an R_n ($e = 1, \cdots, h = n-m$) and let $\overset{a}{u}$ be a set of n linearly inde-

pendent vectors in R_n. Put

$$\overset{\circ}{b}_a = \overset{\circ}{h}_{a(e)} \overset{e}{u}, \quad R_{ab} = R_{ab(e)(d)} \overset{e}{u} \overset{d}{u}$$

and denote by $\overset{\circ}{b}_a \overset{\circ}{b}_e$ the Grassmann products of the b's. Then the Gauss equation for the V_m in R_n may be written

(*) $$R_{ac} = \sum_{e=1}^{h} \overset{\circ}{b}_a \overset{\circ}{b}_c.$$

(2) Denote by $R_{a_1 \cdots a_k, b_1 \cdots b_k}$ the expression obtained from the "determinant" $|R_{a_i b_j}|$ $(i, j = 1, \cdots, k)$ by expanding it without the rule of signs [so that for instance all the summands (Grassmann products) have the sign "plus"]. Using the algebraic properties of Grassmann products, one finds easily that for $n \geq 2h + 2$ a necessary condition for the existence of b's which satisfy (*) is $R_{a_1 \cdots a_{h+1}, b_1 \cdots b_{h+1}} = 0$. Hence these are also necessary algebraic conditions (in the case $n \geq 2h + 2$) for our V_m to be of class h, expressed by means of its curvature tensor. (3) The author generalizes this result also for $n = 2h + 1$ and $n = 2h$. [For the case $h = 1$, cf. T. Y. Thomas, Acta Math. 67, 169–211 (1936); E. Koczian, Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 98, 27–42 (1939).] *V. Hlavatý* (Bloomington, Ind.).

Sen, R. N. Parallel displacement and scalar product of vectors. II. Bull. Calcutta Math. Soc. 41, 41–46 (1949).

This paper is a continuation of a previous paper [Proc. Nat. Inst. Sci. India 14, 45–52 (1948); these Rev. 10, 479] concerned with the parallel displacement of a vector in a Riemannian space, defined by means of arbitrary coefficients of connection Δ^i_{pq}. Additional formal relationships are found connecting Levi-Civita parallelism and the parallelism defined by the Δ^i_{pq}. *A. Fialkow* (Brooklyn, N. Y.).

Kuiper, N. H. On conformally-flat spaces in the large. Ann. of Math. (2) 50, 916–924 (1949).

After some preliminary remarks about conformally flat spaces the author introduces the "$(n+2)$-cyclic" coordinates $X^i / X^0 = x^i$, $X^{n+1} / X^0 = \frac{1}{2} g_{ij} x^i x^j$. The arithmetic n-dimensional space of the x's is transformed by this transformation into an (arithmetic) projective $(n+1)$-dimensional space of the X's on $Q \equiv g_{\lambda \mu} X^\lambda X^\mu = g_{ij} X^i X^j - 2 X^0 X^{n+1} = 0$ $(X^0 \neq 0)$. Conformal transformations are induced by projective transformations in the space (X) leaving Q invariant and vice versa. The author uses these results in order to investigate conformal development of a connected conformally flat space over Q as well as spaces admitting the conformal group of transformations. Typical theorem: a simply connected conformally flat space S can conformally be developed over the space Q in a unique way (except for conformal transformation of Q). The paper ends with some examples of conformally flat spaces (normal space, compact space and so on). *V. Hlavatý* (Bloomington, Ind.).

Rozenfel'd, B. A. The projective differential geometry of the family of pairs $P_m + P_{n-m-1}$ in P_n. Mat. Sbornik N.S. 24(66), 405–428 (1949). (Russian)

This is a development of a considerable amount of work done by the author and others on the application of the metric method to projective, conformal and other geometries. If P_n is an n-dimensional projective space with homogeneous coordinates x^i $(i = 0, 1, \cdots, n)$ and a nondegenerate hyperquadric $a_{\lambda \mu} x^\lambda x^\mu = 0$ is given, the distance ω between two points is taken to be

$$\cos^2 \omega = (a_{\lambda \mu} x^\lambda y^\mu \cdot a_{\lambda \mu} y^\lambda x^\mu) / (a_{\lambda \mu} x^\lambda x^\mu \cdot a_{\lambda \mu} y^\lambda y^\mu)$$

(the radius of the space is taken to be 1; otherwise one would write ω / r for ω). If the above quadric is positive definite the space is elliptic, S_n; if its index is l it is called l-pseudoelliptic, $^l S_n$. If u_λ and v_λ are the tangential coordinates of the polar hyperplanes of x and y $(u_\lambda = a_{\lambda \mu} x^\mu$; $v_\lambda = a_{\lambda \mu} y^\mu)$, the distance is given by

(*) $$\cos^2 \omega = (u_\lambda y^\lambda \cdot v_\mu x^\mu) / (u_\lambda x^\lambda \cdot v_\mu y^\mu),$$

which is a metric invariant of either the two points or the two hyperplanes. It is also a projective invariant of the points and planes as it is the cross-ratio of the four points x, y and the pair of points in which the line xy cuts the two polar hyperplanes. Thus from the projective point of view, the basic element is a point and its polar hyperplane; such an element is called a "0-pair" and (*) is then a projectively invariant distance between two "0-pairs."

Analogously to the complex generalization of S_n one can construct a double unitary elliptic space B_n by using a Clifford algebra: $\alpha = a + eb$, $e^2 = 1$, $\bar{a} = a - eb$, or by changing the basis $(e_1 = (1+e)/2, e_2 = (1-e)/2, e_1^2 = e_1; e_2^2 = e_2; e_1 e_2 = 0)$. In this space the distance between two points is

$$\cos^2 \omega = (\xi^\lambda \bar{\eta}_\lambda \cdot \eta^\lambda \bar{\xi}_\lambda) / (\xi^\lambda \bar{\xi}_\lambda \cdot \eta^\lambda \bar{\eta}_\lambda).$$

The points of B_n may be put into one-to-one reciprocal correspondence with the 0-pairs of P_n by putting $\xi^\lambda = x^\lambda e_1 + u_\lambda e_2$. Multiplication of ξ^λ by $\rho = r_1 e_1 + r_2 e_2$ leaves (*) invariant. It follows then that the group of projective transformations of P_n is simply isomorphic to the group of motions of B_n, the collineations corresponding to proper and correlations to improper motions of B_n (i.e., to motion and $\xi^\lambda \rightarrow \bar{\xi}^\lambda$). By an "$m$-pair" is understood a pair of linear subspaces P_m and P_{n-m-1} such that P_m contains the points of the 0-pairs and P_{n-m-1} is contained in the hyperplanes of the 0-pairs. In general 0-pairs as well as m-pairs consist of nonintersecting elements. Degenerate 0-pairs of P_n for which the point lies in the polar hyperplane are given by $u_\lambda x^\lambda = 0$. Their correspondents in B_n are given by $\xi^\lambda \bar{\xi}_\lambda = 0$, i.e., they are points of the absolute of B_n. (The basis of the theory of 0-pairs and 1-pairs in P_3 is due to É. Cartan; a projectively invariant metric may be introduced in P_3 so that P_3 is a symmetric space with respect to it.)

Since the components of an m-pair are dual one considers only the case $m < n/2$. The essential properties of m-pairs are obtainable from the idea of symmetry with respect to an m-pair. It is a projective involution all of whose invariant points coincide with the points of the m-pair. The problem of finding all involutory elements of the projective group is one of simple Lie groups which was solved by Cartan and in particular by Gantmacher [Rec. Math. [Mat. Sbornik] N.S. 5(47), 217–250 (1939); these Rev. 2, 5]. There are four types of symmetry in P_n: (1) symmetry of real m-pairs, (2) complex conjugate m-pairs in P_{2n+1}, (3) polarities in P_n and (4) null systems in P_{2n+1}. Reciprocal parameters of two m-pairs are determined by symmetries in these m-pairs. Corresponding to these parameters there are invariant subspaces which depend on the characteristic roots and elementary divisors of the matrix defining the product of the two symmetries. If $e^{\omega_1 + i \omega_2}$ is such a root $(\omega_1, \omega_2$ real) so is $e^{\pm \omega_1 \pm i \omega_2}$. Invariant points corresponding to these roots are called director points. The line joining the points α and α^{-1} is symmetric with respect to m-pairs and cuts all four subspaces of the two pairs. These lines are the directrices. If $n > 2m + 1$ and all the characteristic roots are different, two m-pairs have $m + 1$ directrices and $m + 1$ projective invariants $\omega_a = i^{-1} \log \alpha_a$, such that the cross ratio W_a of the four points of intersection of the ath directrix with the four

spaces of the two m-pairs is given by $W_a = -\tan^2 \omega_a$. If the elementary divisors of the matrix are simple, then one m-pair and these reciprocal parameters determine the other m-pair uniquely. Since the group of projective transformations of P_n is a simple noncompact Lie group, one may introduce the invariant metric of Cartan with respect to which P_n is a pseudo-Riemannian symmetric space. The distance ω between two transformations a and b is given in terms of the characteristic roots $\alpha_a = e^{i\omega_a}$ of the matrix $b^{-1}a$ (normalized by the requirement of unimodularity) by $\omega^2 = \omega_0^2 + \omega_1^2 + \cdots \omega_n^2$. The role of geodesics is played by one-parameter subgroups and as Cartan has shown all involutory elements of the group ($a^2 = e$) are conjugate to each other and form a totally geodesic subspace of the group space. The author discusses local parameters of families of m-pairs by means of which the differential geometry of m-pairs is developed. The theory of congruences and pseudocongruences is also treated and families of m-pairs polar with respect to a given $(n-1)$- or $(n-2)$-quadric are discussed (they form totally geodesic hypersurfaces in P_n). It is pointed out that the geometry of m-spheres in C_n (Möbius space) is included in the theory of m-pairs. Finally the author gives a real metric model of the space of m-pairs which is analogous to his construction for a unitary elliptic space.   *M. S. Knebelman* (Pullman, Wash.).

Varga, O. Über affinzusammenhängende Mannigfaltigkeiten von Linienelementen insbesondere deren Äquivalenz. Publ. Math. Debrecen 1, 7–17 (1949).

The paper develops the theory of affine connections for the space of line elements of a manifold. Parallel displacement, covariant differentiation, and torsion and curvature tensors are studied. A solution of the equivalence problem is formulated, expressed naturally in terms of the torsion and curvature tensors and their successive covariant derivatives.   *S. Chern* (Chicago, Ill.).

Vranceanu, Georges. La géométrisation des espaces symplétiques. C. R. Acad. Sci. Paris 229, 336–338 (1949).

The paper studies spaces with a given exterior quadratic differential form and establishes that in certain cases the spaces are "geometrisable" in the sense of the author.   *S. Chern* (Chicago, Ill.).

Vranceanu, Georges. Sur les espaces à connexion à groupe maximum des transformations en eux-mêmes. C. R. Acad. Sci. Paris 229, 543–545 (1949).

The following theorem is proved. An n-dimensional affinely or projectively connected space whose projective curvature tensor is different from zero admits a group with at most $n^2 - 2n + 5$ or $n^2 - 3n + 8$ parameters, according as the components of the curvature tensor γ^a_{bcd} are all zero or not, for a, b, c, d distinct. Examples are given showing that the maxima are attained.   *S. Chern* (Chicago, Ill.).

Vagner, V. V. The theory of a field of local hyperstrips in X_n and its application to the mechanics of a system with nonlinear anholonomic connection. Doklady Akad. Nauk SSSR (N.S.) 66, 1033–1036 (1949). (Russian)

An $(m-1)$-dimensional hyperstrip in a centro-affine space E_n is an $(m-1)$-dimensional surface to every point of which is attached a tangent hyperplane. Several ways are mentioned in which such a hyperstrip can be characterized analytically in terms of affine connections, affinors and tensors. If in every tangent E_n to an X_n a hyperstrip is given the author speaks of a field of local hyperstrips;

together with the X_n they form a combined manifold X_{n+m-1} [cf. C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 94–97 (1943); these Rev. 6, 106]. In this manifold a linear connection can be given in an invariant way, which, under certain conditions, permits one to introduce a parallel displacement, an absolute derivative, and a curvature vector for certain curves (called measurable). For $m = n$ the above theory reduces to the Finsler geometry, and the displacement to that of Berwald. The preceding theory is applied to the discussion of motion of a mechanical system with n degrees of freedom (considered as the motion of a point of unit mass in a Riemannian V_n whose metric tensor is essentially the kinetic energy) with $n - m$ nonlinear, nonholonomic constraints involving velocities. At every point of the V_n the directions satisfying the constraints form an m-dimensional surface in the tangent E_n; as the basic surface the author takes the intersection of this surface with the hyperellipsoid whose equation is obtained by equating to 1 the square of the linear element; the hyperplane attached to every point is the hyperplane to the ellipsoid. The equations of motion are written in terms of the above geometrical theory.   *G. Y. Rainich* (Ann Arbor, Mich.).

Liber, A. E. On spaces of linear affine connection with one-parameter holonomy groups. Doklady Akad. Nauk SSSR (N.S.) 66, 1045–1046 (1949). (Russian)

Given a linearly connected L_n with components of linear connection $\Gamma^\lambda_{\mu\nu}(\xi^a)$; if x^λ are the coordinates of a point in the tangent space the solutions of $dx^\lambda/dt + \Gamma^\lambda_{\mu\nu} x^\nu d\xi^\mu/dt = 0$ determine a mapping of one tangent space upon another, defined by the curve $\xi^a(t)$. The totality of mappings of the tangent space at P upon itself, determined by all closed curves through P, constitutes the holonomy group of the space L_n since the groups at different points are similar and one is carried into the other by a central affine transformation. The holonomy group is thus either a central affine group or a subgroup of it. If the group is the identity then the curvature of L_n is zero. The paper is concerned with the case of linearly connected spaces whose holonomy group depends on only one parameter. In this case the vector of the infinitesimal transformation is $v^\lambda = C_\mu^\lambda x^\mu$. Since the holonomy group has $n-1$ independent invariants L^p ($p = 1, \cdots, n-1$) which determine the one-parameter group uniquely, the group which leaves C_μ^λ invariant coincides with the group that leaves L^p invariant. Thus in the space L_n there must exist $n-1$ scalars L^p whose covariant derivatives $D_\mu L^p = 0$. This gives $R^a_{\mu\nu\lambda} = V_{\nu\mu} C_\lambda^a$; $\nabla_\nu C_\lambda^a = 0$, where $V_{\nu\mu}$ is a bivector. If the components of linear connection are symmetric, the rank of C_μ^λ is at most 2 so that for a Riemann space with a one-parameter holonomy group there exist $n-2$ parallel vector fields. Thus the only Riemann spaces with one-parameter holonomy groups are products of a two-dimensional Riemann space and an $(n-2)$-dimensional Euclidean space.   *M. S. Knebelman*.

Akivis, M. A. A focal family of rays as the image of a pair of T-complexes under a Plücker transformation. Doklady Akad. Nauk SSSR (N.S.) 65, 429–432 (1949). (Russian)

Continuation of a previous paper [same Doklady 61, 181–184 (1948); these Rev. 10, 400]. A pair of T-complexes is embeddable in a five-dimensional projective space and by means of a Plücker quadric a three-dimensional surface S is associated with every pair of T-complexes. The paper gives a number of geometric properties of the configuration consisting of T and S.   *M. S. Knebelman*.

Norden, A. P. The space of linear congruences. Mat. Sbornik N.S. 24(66), 429–455 (1949). (Russian)

This paper is closely related to earlier papers by the author [in particular, C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 195–197 (1947); Doklady Akad. Nauk SSSR (N.S.) 58, 1597–1599 (1947); these Rev. 8, 604; 9, 467], of which it seems to be an elaboration. A geometry K is considered whose group consists of projective transformations leaving invariant an elliptic congruence. This congruence plays the role of the absolute; its lines are called improper lines. Parallelism and angular measure are introduced; through every point passes a pencil of lines parallel to a given line. A spheroid consists of all improper lines meeting a given proper line; a circle is the section of a spheroid by a plane. A natural correspondence between two planes is established by assigning to each other points on the same improper line; circles go into circles in this correspondence. By a proper choice of a reference tetrahedron a point may be characterized by two complex coordinates homogeneous in the sense that multiplication of both by a real number does not alter the point. Transformations of these coordinates are given by two-by-two complex matrices determined up to a real factor. These matrices also represent the group of K, which thus has seven parameters. Bicylinders are quadric surfaces with two families of rectilinear generators, two generators of the same family being parallel, and two of different families, orthogonal; the axes of the two (imaginary) special complexes are conjugate with respect to a bicylinder. The inner geometry of any surface in K is, according to the terminology introduced in earlier papers, quasi-Euclidean (that of a plane, Euclidean); parallelism and angular measure in that geometry coincide with parallelism and angular measure in K. A surface can be mapped onto a plane by improper lines; this mapping is conformal and defines on the plane a scalar field φ which, in turn, determines the connection of the inner geometry of the surface. An arbitrary abstract quasi-Euclidean geometry may be imbedded in K. A surface is said to be of zero curvature if its Riemann tensor is zero. A necessary and sufficient condition for this is that the scalar field φ is harmonic. By the mapping of K on the complex plane an analytic function, which may be interpreted as a curve of that plane, corresponds to a surface of zero curvature of K. The two quadratic forms of the inner geometry of that surface are essentially the real and imaginary parts of the Schwartzian differential. To a fractional linear function corresponds a plane; to the logarithm, a bicylinder; in a certain sense the surfaces of zero curvature of K can take the place of Riemann surfaces of the corresponding analytic function. *G. Y. Rainich* (Ann Arbor, Mich.).

Rapcsák, A. Kurven auf Hyperflächen im Finslerschen Raume. Hungarica Acta Math. 1, no. 4, 21–27 (1949).

Consider an arbitrary curve: $y^\alpha = y^\alpha[x^i(s)]$ ($\alpha=1, \cdots, n$; $i=1, \cdots, n-1$) on a hypersurface in an n-dimensional Finsler space; then the curve has $n-1$ invariants (i.e., curvatures) ρ_n^{-1} (putting $\rho_n^{-1}=0$) as a space curve but $n-2$ invariants μ_i^{-1} (putting $\mu_{n-1}^{-1}=0$) as a curve on the hypersurface. In order to find the relations between these systems of invariants, the author considers the two intrinsic orthogonal n-ples $t^\alpha_{(\beta)}$ consisting of the tangent and $n-1$ normals as a space curve and $k^\alpha_{(i)}$ ($(n-1)$-ple k^α as a curve on the hypersurface and $k^\alpha_{(n)}=$surface normal), between which the relations $k^\alpha_{(\gamma)}=c_{\beta\gamma} t^\alpha_{(\beta)}$ hold good; $c_{\beta\gamma}=\cos(k_{(\beta)}, t_{(\gamma)})$. Comparing two systems of Frenet formulas for $t^\alpha_{(\beta)}$ and $k^\alpha_{(\beta)}$, the author derives the relations between the two systems of invariants, ρ's and μ's, in terms of $c_{\beta\gamma}$ and their derivatives. These relations contain formulas which can be regarded as generalization of the Darboux relations for the motion of the moving trihedra of a surface. It should be remarked that the author uses as the absolute differential on the hypersurface the one induced orthogonally from the absolute differential in the space, following O. Varga [Deutsche Math. 6, 192–212 (1941); these Rev. 8, 231]. At the end it is shown that some well-known geometrical theorems (e.g., a geodesic line in the space is also a geodesic line on the hypersurface, etc.) can be derived easily from these relations. There are some misprints. *A. Kawaguchi*.

NUMERICAL AND GRAPHICAL METHODS

✱**Tables of the Bessel Functions of the First Kind of Orders Fifty-Two Through Sixty-Three. By the Staff of the Computation Laboratory.** Harvard University Press, Cambridge, Mass., 1949. ix+544 pp. $8.00.

The present volume contains tables of $J_n(x)$ for $n=52(1)63$ and $x=0(.01)99.99$ to 10 decimal places. *A. Erdélyi*.

Laurent, Mariette. Table de la fonction elliptique de Dixon pour l'intervalle 0—0, 1030. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 439–450 (1949).

In considering suitable projections for a map of the Belgian Congo the conformal representation of the sphere in an equilateral triangle appeared to be advantageous. Existing tables for this projection [O. S. Adams, Elliptic Functions Applied to Conformal World Maps, U. S. Coast and Geodetic Survey, Special Publication, no. 112, Washington, D. C., 1925] were inadequate. The function involved is the Dixon elliptic function sm u. The main body of this paper is a very detailed table giving sm u for $u=0(.0010).1030$, 10 D, $\Delta_1, \Delta_2, \Delta_3$, and the inverse table of u for sm $u=0(.0010).1030$, 10 D, $\Delta_1, \Delta_2, \Delta_3$. *N. A. Hall*.

Byhovskiǐ, M. L. Principles of electronic mathematical machines for discrete calculation. Uspehi Matem. Nauk (N.S.) 4, no. 3(31), 69–124 (1949). (Russian)

This is an expository article on components of electronic digital computers. No one machine is described or even referred to; only fundamental components are treated. The article is well supplied with line drawings and a few photographs. The presentation is clear and straightforward and apparently regarded as sufficiently complete since the article contains no references to other works from which the material was extracted. Nothing new is contributed to the subject. The only major component apparently unknown to the author is the magnetic drum memory. Detailed descriptions (including values of resistors, etc., and tube numbers) are given of the following components: electronic static memories; ring counters and adders; memories: punch cards, paper tapes, selectron, cathode ray tubes, mercury delay lines; multipliers; control systems; pulse generators; punch card in- and output. *D. H. Lehmer*.

Birge, Raymond T. The exact representation of a series of points by a polynomial in power series form. Amer. J. Phys. **17**, 196–200 (1949).

A simple algorithm is given for obtaining, in power series form, the polynomial of degree n fitted to $n+1$ given values of a function.   *T. N. E. Greville* (Washington, D. C.).

Joseph, A. W. A comment on interpolation in two variables. J. Inst. Actuar. **74**, 82–85 (1948).

The author gives a geometric interpretation of the usual elementary method of interpolation in two variables in which a polynomial of specified degree in x and y together is fitted to values of $f(x, y)$. He shows that the number of given values $f(x_i, y_i)$ required can be reduced somewhat by choosing the points (x_i, y_i) so that a number of them lie on the same curve (of the specified, or lower, degree) with the point (x, y) for which an interpolated value is desired.
    T. N. E. Greville (Washington, D. C.).

Markovitch, D. Sur quelques formules approchées pour la racine carrée d'un nombre. Bull. Soc. Math. Phys. Serbie **1**, 71–76 (1949). (Serbian. Russian and French summaries)

Let $x^2 = N$. The author iterates the identity

$$x = (N + \lambda x)/(\lambda + x).$$

Substituting various values for λ (such as $\lambda = [N^{\frac{1}{2}}]$) he obtains a collection of approximations to x.   *W. Feller.*

Bodewig, E. On types of convergence and on the behavior of approximations in the neighborhood of a multiple root of an equation. Quart. Appl. Math. **7**, 325–333 (1949).

This is an English translation of a recent paper by the author [Z. Angew. Math. Mech. **29**, 44–51 (1949); these Rev. **10**, 573].   *E. Frank* (Chicago, Ill.).

Reich, Edgar. On the convergence of the classical iterative method of solving linear simultaneous equations. Ann. Math. Statistics **20**, 448–451 (1949).

Ist die Matrix A des Gleichungssystems reell, symmetrisch und hat lauter positive Diagonalelemente, so gilt bekanntlich der Satz, dass die Seidelsche Methode konvergiert, wenn A positiv definit ist. Verfasser beweist nun, dass die Bedingung der Definitheit von A auch notwendig ist zur Konvergenz des Seidel-Prozesses.   *E. Bodewig.*

Kohn, Walter. A variational iteration method for solving secular equations. J. Chem. Phys. **17**, 670 (1949).

Es sei E ein Eigenwert der symmetrischen Matrix A, und E_0 ein Näherungswert. Dann setze man eine Komponente, etwa $x_k = 1$ und löse das Gleichungssystem $Ax = E_0x$ unter Weglassen der k-ten Gleichung. Dann erhält man einen besseren Näherungswert E_1 aus $E_1 = x'Ax/x'x = E_0 + (E' - E_0)/x'x$, wo $E' = \sum a_{kj}x_j$. Das Verfahren konvergiert quadratisch.
    E. Bodewig (Den Haag).

Murray, F. J. Linear equation solvers. Quart. Appl. Math. **7**, 263–274 (1949).

This article considers automatic adjusting types of linear simultaneous equation solvers where the process is either continuous as in an analogue equation solver, or in discrete steps as in a digital computing routine. If $\epsilon_i = \sum_{j=1}^n a_{ij}x_j + b_i$, the only adjusting processes considered are assumed to be linear operators on the ϵ_i. "In the present article, we point out that if an adjusting type of machine is to operate successfully whenever the determinant A is not zero, then

the square of the determinant must enter the indicial equation of the equations of motion for the machine. This necessary condition for successful operation rules out any linear feedback which does not involve using the a_{ij} twice. This result generalizes certain aspects of the necessity argument indicated" by Goldberg and Brown [J. Appl. Phys. **19**, 339–345 (1948); these Rev. **9**, 535].   *R. Hamming.*

Korn, Granino A. Stabilization of simultaneous equation solvers. Proc. I.R.E. **37**, 1000–1002 (1949).

This paper is concerned primarily with simultaneous linear equation solving machines in which the error in the ith equation is used to change the ith unknown directly through a feedback amplifier. It is assumed that the system of equations is positive definite. The problem of designing the amplifiers so that a system of n equations will be stable is reduced to that of designing an amplifier for a single equation $ax + b = 0$, where a may be a complex number with a positive real part. "The fundamental importance of the last-mentioned theorem lies in the fact that it reduces the design of amplifiers for the complicated multiple-loop feedback system to the design of one simple feedback amplifier, so that stability criteria known from experience or a Nyquist analysis may be applied. The analysis described can be extended to cases in which the various amplifiers are not exactly identical as has been assumed above. Other more general theorems useful for the design of multiple-loop feedback amplifiers and servo-mechanisms may be derived."
    R. Hamming (Murray Hill, N. J.).

Lyusternik, L. A. On electrical modelling of symmetric matrices. Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 198–200 (1949). (Russian)

H. Bode has shown [Z. Angew. Math. Mech. **17**, 213–223 (1937)] how the unknowns of a system $\sum_{j=1}^n a_{ij}x_j = m_i$, $i = 1, \cdots, n$, can be represented by the potentials of points $A_1, \cdots, A_n$ (current sources equal to the m_i) relative to the point A_0 if these $n+1$ nodes are interconnected by resistances (determined by the a_{ij}) provided that for the matrix of the system

$$(1) \quad a_{ij} = a_{ji}, \quad (2) \quad a_{ii} \geqq \sum_{j \neq i} |a_{ij}| \quad \text{and} \quad (3) \quad a_{ij} \leqq 0, \quad j \neq i.$$

It is shown that by using n additional points $B_1, \cdots, B_n$, the potential of B_i relative to A_0 to be the negative of that of A_i, the last condition can be removed. This is done by writing the left hand sides of the equations as

$$\sum_{j \neq i} |a_{ij}| (x_i + x_j \operatorname{sgn} a_{ij}) + (a_{ii} - \sum_{j \neq i} |a_{ij}|)x_i.$$

Then A_i is connected to B_j or A_j and B_i is also connected to A_j or B_j, $j \neq i$, by conductances equal to a_{ij} according as $\operatorname{sgn} a_{ij}$ is plus or minus one, and A_i and B_i are each connected to A_0 by conductances equal to $a_{ii} - \sum_{j \neq i} |a_{ij}|$. The principle of superposition can be employed to advantage in almost exactly the same way as was done by Bode to make only one current source necessary.   *R. Church.*

Puwein, Max Georg. Eine Quadraturformel. Anz. Akad. Wiss. Wien. Math.-Nat. Kl. **83**, 25–26 (1946).

Durch Integration der Besselschen Interpolationsformel erhält der Verfasser die Formel:

$$\int_a^b f(x)dx \approx S - h(\tfrac{1}{2}y_0 + \tfrac{1}{2}y_n - \tfrac{1}{2}y_1 - \tfrac{1}{2}y_{n-1} + \tfrac{1}{3}y_2 + \tfrac{1}{3}y_{n-2}),$$

wo $S = h(\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n) = $ Sehnenpolygon, $h = (b-a)/n$. Dabei ist vorausgesetzt, dass die dritten Differenzen verschwinden.   *E. Bodewig* (Den Haag).

Puwein, Max Georg. Eine Rektifikationsformel. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **84**, 77–79 (1947).

Verfasser gibt eine von der numerischen Quadratur unabhängige Rektifikationsformel. Danach ist die Länge der Kurve, von der die Punkte $0, 1, \cdots, n$ gegeben sind, ungefähr gleich $\frac{1}{3}\sum_{k=1}^{n} s_{k-1, k} - \frac{1}{12}A$, wo

$$A = s_{01} + s_{n-1, n} - s_{12} - s_{n-2, n-1} + s_{02} + s_{n-2, n} + 2\sum_{k=2}^{n} s_{k-2, k}$$

und s_{pq} die Sehne zwischen den Punkten p und q ist.
$E.\ Bodewig$ (Den Haag).

Ascoli, Guido. Vedute sintetiche sugli strumenti integratori. Rend. Sem. Mat. Fis. Milano **18** (1947), 36–54 (1948).

This is an attempt at a unified theory of certain planimeters and integraphs. A planimeter is defined as a mechanical analogue device for determining a double integral over the area enclosed by a curve by tracing the curve with a pointer. Such a device is necessarily a nonholonomic mechanical system, and therefore contains as constituent at least a rolling wheel. The theory of an instrument containing such a rolling wheel is discussed under rather general assumptions and then specialized. The key tool is Green's theorem for a plane area. There is a briefer discussion of integraphs, i.e., devices for drawing the graph of $\int_a^x f(x)dx$ by tracing that of $f(x)$. $H.\ B.\ Curry$ (State College, Pa.).

Walz, Alfred. Ein waageähnliches Gerät für harmonische Analyse und Synthese. Arch. Math. **1**, 383–392 (1949).

A device for harmonic analysis is described which utilizes the Bessel summation formulas for numerical harmonic analysis by providing for introducing the $2n$ measured ordinates of the curve to be analyzed as the positions of unit weights on $2n$ levers which are capable of rotation around a common horizontal axis. For each Fourier coefficient obtainable there is a curved template with stairstep profile by means of which the levers can quickly be rotated close to the appropriate angles with respect to the vertical and then by turning a long bolt of oval cross-section through 90° they are locked accurately in correct relative position. The static moment about the axis then provides evaluation of the Bessel formula. Pictures of a preliminary model for 24 ordinates are shown. The moment principle has previously been suggested by various writers; the present design is advanced as a practical embodiment which at the same time avoids some of the disadvantages of planimeter type analyzers. The author believes greater speed of operation and comparable accuracy obtainable with such machines of moderate size if constructed with precision. In addition they can be used for point-by-point harmonic synthesis.
$R.\ Church$ (Annapolis, Md.).

✱von Sanden, Horst. Praxis der Differentialgleichungen. Eine Einführung. 3d ed. Walter de Gruyter & Co., Berlin, 1945. 105 pp. 5 DM.

This book is divided into two distinct parts. Part A treats graphical and numerical solutions of ordinary differential equations where the desired solution is completely determined at the starting point. The graphical solution of a first order equation is sketched by means of the isoclinal curves. In the step-by-step numerical integration of a first order equation the author predicts y_{k+1} by the formula

$$y_{k+1} = y_k + h\left[1 + \tfrac{1}{2}\Delta + \tfrac{5}{12}\Delta^2 + \tfrac{3}{8}\Delta^3 + \tfrac{251}{720}\Delta^4 + \cdots\right]y_k'$$

(in which Δ represents the backward difference operator) and corrects by the formula

$$y_{k+1} = y_k + h\left[1 - \tfrac{1}{2}\Delta - \tfrac{1}{12}\Delta^2 - \tfrac{1}{24}\Delta^3 - \tfrac{19}{720}\Delta^4 - \cdots\right]y_{k+1}'.$$

An estimate of the error is provided. Brief mention is made of analytic iteration and of the Runge-Kutta method. In a like manner he deals with second order equations, first giving a graphical solution, and then a step-by-step process using suitable difference formulas. The treatment is then extended to systems of equations of first and second orders.

Part B deals with boundary value problems for the case of second order linear differential equations containing a parameter, where the solution must satisfy conditions at two points. Both homogeneous and nonhomogeneous cases are considered. Three approximate methods of finding characteristic numbers are presented, that of Rayleigh, a method of iteration, and the Ritz method. $W.\ E.\ Milne.$

✱Collatz, Lothar. Eigenwertaufgaben mit technischen Anwendungen. Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 19. Akademische Verlagsgesellschaft, Leipzig, 1949. xvii+466 pp.

This book is identical with the author's earlier publication entitled "Eigenwertprobleme und ihre numerische Behandlung" [Akademische Verlagsgesellschaft, Leipzig, 1945; these Rev. **8**, 514], except for a slight rearrangement and the inclusion of an extensive new chapter called "Eigenwertaufgaben bei Matrizen." Since the earlier book was rather completely reviewed this review will cover only the added chapter.

This chapter begins with a discussion of matrices and their latent roots, quadratic and Hermitian forms, reality of characteristic roots, together with illustrative examples chosen from mechanics. Then follows a discussion of the maximal properties of the characteristic numbers, transformation of the principal axes, Courant's maximum-minimum principle, the "Einschliessungssatz," and various numerical applications. Next comes a treatment of the process of iteration, the principal vectors, bounds for the characteristic numbers, convergence of the iteration process, and applications to systems of ordinary linear differential equations. The chapter concludes with a consideration of matrix polynomials and Cayley's equation, matrix functions and power series, approximate solutions of systems of linear equations, and special methods for setting up the characteristic equation. $W.\ E.\ Milne$ (Corvallis, Ore.).

Babkin, B. N. On a modification of S. A. Čaplygin's method for approximate integration. Doklady Akad. Nauk SSSR (N.S.) **67**, 213–216 (1949). (Russian)

The author considers the first order differential equation $y' = f(x, y)$. The method of Čaplygin presupposes that f and $\partial f/\partial y$ are continuous inside a region V_1 while $\partial^2 f/\partial y^2$ does not change sign there. The author proceeds to show that if one modifies the method of Čaplygin suitably the condition imposed upon $\partial^2 f/\partial y^2$ becomes unnecessary. The initial condition to be satisfied is taken as $y(0) = y_0$. Suppose that two functions $z = z(x)$ and $t = t(x)$ have been found, both of them going through $A(x_0, y_0)$ and lying entirely in v. Assume that $dz/dx - f(x, z) < 0$ and $dt/dx - f(x, t) > 0$; then, according to the theorem of Čaplygin, $z(x) < y(x) < t(x)$ for $x_0 < x \leq x_1$. Introducing $\alpha(x) = dz/dx - f(x, z)$ and $\beta(x) = dt/dx - f(x, t)$ as well as σ_1 and η_1 defined by

$$\frac{\partial \sigma_1}{\partial x} - m\sigma_1 - \beta(x) = 0; \quad \frac{d\eta_1}{dx} - m\eta_1 + \alpha(x) = 0$$

(where it is clear that $\sigma_1(x) > 0$ and $\eta_1(x) > 0$), the author proceeds to prove that:

$$\sigma_1(x) < t(x) - y(x); \quad \eta_1(x) < y(x) - z(x),$$

i.e., $y(x) < t(x) - \sigma_1(x)$ and $z(x) + \eta_1(x) < y(x)$. Taking $t(x) - \sigma_1(x) = t_1(x)$ and $z(x) + \eta_1(x) = z_1(x)$ a new set of approximations is obtained for the solution $y = y(x)$. The author shows that $dz_1/dx - f(x, z_1) < 0$; $dt_1/dx - f(x, t_1) > 0$. Similarly η_2 and σ_2 are introduced through

$$\partial \sigma_2/\partial x - m \sigma_2 - \beta_1(x) = 0$$

and

$$d\eta_2/dx - m\eta_2 + \alpha_1(x) = 0$$

and a set of approximations $\{t_n(x)\}$ and $\{z_n(x)\}$ obtained between which the true solution $y(x)$ is found to lie. The case $m = 0$ corresponds to Picard's method.

By solving the defining linear differential equations for $\sigma_1(x)$ and $\eta_1(x)$ it is shown that $\{\sigma_n(x)\}$ and $\{\eta_n(x)\}$ converge uniformly toward zero, from which fact there follows, in turn, the uniform convergence of $\{t_n(x)\}$ and $\{z_n(x)\}$.

The rapidity of convergence can be increased somewhat by making the coefficient in the equations above depend upon the value of the index n, thus:

$$\partial\sigma_n/\partial x - m_{n-1}\sigma_n - \beta_{n-1}(x) = 0; \quad \partial\eta_n/\partial x - m_{n-1}\eta_n + \alpha_{n-1}(x) = 0,$$

where m_{n-1} is chosen so as to make $m_{n-1} < \partial f/\partial y$ ($m_{n-1} \geqq m_{n-2}$) in the suitably defined region $\bar{v}_{n-1}$ (that is, bounded by the curves $t_{n-1} = t_{n-1}(x)$, $z_{n-1} = z_{n-1}(x)$ and the line $x = x_1$).

M. Daniloff (Cambridge, Mass.).

Lachmann, Kurt. Ein neues Verfahren zur Lösung von Randwertaufgaben bei gewöhnlichen Differentialgleichungen zweiter Ordnung. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1946**, no. 3, ii+38 pp. (1946).

The object of the paper is to obtain the solutions of the differential equation $y'' + \varphi(x, y, y') = 0$ which go through the points (x_1, y_1), (x_2, y_2), with $x_1 \neq x_2$. By a method similar to that of differential variations the author obtains a linear differential equation of second order whose solutions are used to improve trial solutions of the problem. He investigates conditions under which the process converges, and as an illustration applies the method to Duffing's oscillation problem. *W. E. Milne* (Corvallis, Ore.).

Lahaye, Edmond. Une méthode de résolution des équations différentielles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 851–862 (1948).

The author generalizes the differential equation $F = 0$ by introducing a parameter t satisfying certain conditions, but

so that for $t = 1$ the original equation arises. The new equation $\phi = 0$ is solved for $t = 0$ yielding the solution y_1. The expansion of $\phi = 0$ in powers of $y - y_1$, $y' - y_1'$, $\cdots$ is broken off after the linear terms and the new (linear) equation $\phi_1 = 0$ is solved by successive approximations. Then t is increased successively and every new equation is solved, until t has become 1. *E. Bodewig* (The Hague).

Metropolis, Nicholas, and Ulam, S. The Monte Carlo method. J. Amer. Statist. Assoc. **44**, 335–341 (1949).

The authors present the motivation and general description of a method which deals with certain problems by identifying their solutions with those of corresponding probability problems. The latter solutions are to be found approximately by sampling. *J. L. Doob* (Ithaca, N. Y.).

Salonen, Eero. Über die Formeln für die Fehlergrenzen der Polygonmessung. Schweiz. Z. Vermessg. Kulturtech. **47**, 235–247 (1949).

Mazzoni, Pacifico. La formola di Hattendorf generalizzata ed il rischio medio. Atti Ist. Naz. Assicuraz. **12**, 45–64 (1940).

The author starts with the statement that Hattendorff's well-known theorem is not valid for a pure endowment insurance. He gives an example and uses there a formula due to J. F. Steffensen [Skand. Aktuarietidskr. **12**, 1–17 (1929)]. However, Steffensen's formula was derived only for whole life insurances. The indiscriminate use of this formula is the reason for the author's belief that Hattendorff's formula is not applicable in his example. The difficulty is resolved as soon as the mean risk is defined as the expected value of the squares of the losses (or profits). The author then extends Hattendorff's formula; similar or even more far reaching generalizations were given by A. Berger [Mathematik der Lebensversicherung, Springer, Wien, 1939, in particular, § 57; Trans. Twelfth Internat. Congress Actuaries, Lucerne, 1940, vol. IV, pp. 9–26; these Rev. 3, 177] and by the reviewer [ibid., vol. I, pp. 171–205; these Rev. 3, 177]. *E. Lukacs* (China Lake, Calif.).

Mazzoni, Pacifico. Intorno al metodo d'interpolazione del Lever. Atti Ist. Naz. Assicuraz. **14**, 163–179 (1942).

The paper deals with the problem of computing the value of a whole life annuity at a given rate of interest if the value of this annuity is known for two other rates. The author surveys various methods of interpolation and stresses the usefulness of a method suggested by E. H. Lever [J. Inst. Actuar. **52**, 171–179 (1921)]. *E. Lukacs.*

ASTRONOMY

⁕Schütte, Karl. Mathematische Methoden der Astronomie. Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 101–119. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=\$2.40.

The article summarizes the somewhat heterogeneous group of astronomical papers published in Germany between 1939 and 1946 in which use was made of mathematical methods. The special areas covered are: spherical astronomy, theoretical astronomy (cosmology, celestial mechanics), stellar astronomy (statistics, binaries, galactic dynamics) and theoretical astrophysics (planetary atmospheres, comets, stellar rotation, interiors, interstellar matter). The author indicates that the survey is supplementary to

volume 20, on Astronomy, Astrophysics and Cosmogony, of the same collection. *B. J. Bok* (Bloemfontein).

de Orus Navarro, Juan J. On a problem of celestial mechanics. Revista Mat. Hisp.-Amer. (4) **9**, 13–15 (1949). (Spanish)

Using the formulae of classical (nonrelativistic) mechanics, the author obtains, to a first approximation, the secular inequalities of the orbital elements of a planet when the perturbing force is central and proportional to $1/r^n$. He finds that the secular inequalities of the major axis and eccentricity are zero, but that there is a change in the longitude of perihelion. Taking $n = 4$, he is able to deduce

the well-known relativistic formula for the motion of the perihelion of Mercury.   *H. S. Ruse* (Leeds).

Stumpff, K. Über die Beziehungen zwischen den Anomalien der Keplerbewegung. Astr. Nachr. 277, 55–58 (1949).

It is shown that the relations among the eccentricity angle φ, the true anomaly v, and the eccentric anomaly E in elliptic motion correspond to the relations in a spherical triangle the sides of which are φ, $\frac{1}{2}\pi-v$ and $\frac{1}{2}\pi-E$, the opposite angles, φ, $\frac{1}{2}\pi-v$ and $\frac{1}{2}\pi+E$, respectively. In a different triangle the "antifocal" anomaly takes the place of the true anomaly. Similar triangles applicable to parabolic and hyperbolic motion are also presented.

D. Brouwer (New Haven, Conn.).

de Jekhowsky, Benjamin. Sur la détermination du paramètre dans le problème d'Euler. C. R. Acad. Sci. Paris 228, 1925–1927 (1949).

The author presents a modification of a numerical method by H. Andoyer for obtaining the parameter of an orbit from two known heliocentric positions at instants t_1 and t_2. While in Andoyer's method three or four approximations were required, the use of a series in powers of $\tau=k(t_2-t_1)$ is shown to yield the required solution, at least for moderate values of τ, in a single step.   *D. Brouwer* (New Haven, Conn.).

Orlov, B. A. Application of Delaunay-Hill's method to the case of commensurability 3:4 (Thule type). Bull. Observ. Poulkovo 17, no. 3(138), 71–92 (1947). (Russian. English summary)

Delaunay's method in the lunar theory [Mém. Acad. Sci. Paris 28 (1860); 29 (1867)] was modified by Hill [Collected Mathematical Works, v. 4, Washington, D. C., 1907, pp. 169–206] so as to make it applicable to the general problem of planetary motion, and then applied to the cases of commensurability 1:2 (Hecuba type) and 1:3 (Hestia type) in the restricted problem of three bodies by Hill himself and by Åkesson [Ark. Mat. Astr. Fys. 8, no. 25 (1913)], respectively. In the present paper the same method is applied by the author to the case of commensurability 3:4 (Thule type). The disturbing function is developed by Newcomb's method up to the sixth power of eccentricity and for numerical calculations the osculating elements of (279) Thule for the epoch 1925 are used. The main results are the following. (1) The asteroid has a periodic orbit with a period of about 172 Julian years. 2. The libration of the critical argument θ about the value $\theta=0$ cannot exceed 70°, the libration taking place for eccentricities $\varepsilon<0.09$. (3) The perihelion of the orbit has a retrograde motion whose velocity exceeds one degree per year. As the application of Delaunay's method to the restricted problem of three bodies is confined to one operation, it is clear that such solutions give only an approximation of the real motion. [Misprints: read θ for Θ: p. 73 formula (3), second formula of system (4), lines 8 and 12 counted from the bottom; p. 74 first formula on the right and second on the left of system (5), system (7), lines 13, 14 from top and lines 9, 12, 13, 19 counted from the bottom; p. 75 formulas in lines 13, 18, 19; p. 76 line 2; p. 78 line 2.]   *E. Leimanis.*

Bilimovitch, Anton D. Application of Pfaff's method and of vectoral elements to the problem of three bodies. Glas Srpske Akad. Nauka 191, 139–148 (1948). (Serbian. English summary)

The author applies the methods of his previous paper [Astr. Nachr. 273, 161–178 (1943); these Rev. 6, 243] to

the problem of three bodies and derives the classical results. The same analysis has also appeared in a paper by Musen [Z. Naturforschung 3a, 360–363 (1948); these Rev. 10, 577].

R. G. Langebartel (Urbana, Ill.).

Colacevich, A. Relazioni analitiche tra le distribuzioni delle velocità lineari delle stelle. Mem. Soc. Astr. Ital. (N.S.) 18, 177–198 (1 plate) (1947).

The paper also appeared in Osservazioni e Memorie dell'Osservatorio Astrofisico di Arcetri, no. 64, 3–24 (1947); these Rev. 10, 487.

Krat, W. Solar hydrodynamics. I. Bull. Observ. Poulkovo 17, no. 1(136), 58–88 (1947). (Russian. English summary)

The essential mathematical portions of the paper deal with the problem of the rotation of a polytropic gas sphere in which the angular velocity ω has the form

$$\omega^2 = c_0^2\omega_0^2 + \sum_1^\infty c_j\omega_j^2 P_j(\mu),$$

where the c_j's are numerical coefficients and $P_j(\mu)$ is the Legendre polynomial of order j. From the discussion of this case the author concludes that "all barotropic configurations are initially instable for any displacements of barotropic or baroclinic nature." The paper also summarizes at some length the relevant literature on the subject.

S. Chandrasekhar (Williams Bay, Wis.).

Vogt, H. Zur Theorie der Sternrotation. Astr. Nachr. 277, 49–54 (1949).

The author does not seem to be aware of the work done in the field in America during the last ten years, and consequently repeats it to a large extent. In his special calculation, which is new, he starts with the assumption of a mass-point at the center of a star, and calculates the distribution of temperature and density throughout the star, without questioning whether the calculated mass-distribution in any way approximates the central mass-point originally considered.   *G. Randers* (Oslo).

Biberman, L. M. On the equations of radiation transfer in stellar atmospheres. Doklady Akad. Nauk SSSR (N.S.) 67, 443–445 (1949). (Russian)

In problems of radiation transfer in stellar envelopes it is usually assumed that no change in frequency occurs in the process of photon diffusion. As Spitzer has pointed out, this assumption is not rigorously correct: inasmuch as the components of the velocity of the atom in the directions of the absorbed and of the radiated photon do not coincide, there must be a difference in the frequency of the photons. In the present paper an integro-differential equation is derived for the intensity of radiation of a given frequency, in functions of the coordinates and the direction, and an integral equation for the concentration of the excited atoms. In both equations allowance has been made for the above mentioned frequency variation.   *L. Jacchia* (Cambridge, Mass.).

Sobolev, V. V. Incoherent scattering of light in stellar atmospheres. Akad. Nauk SSSR. Astr. Zhurnal 26, 129–137 (1949). (Russian)

The author begins by discussing the problem of the diffuse reflection of radiation from a plane-parallel atmosphere capable of absorption and of noncoherent scattering, and obtains an equation for $I(\nu, \nu_1, \eta, \zeta)$, the intensity of out-

ward radiation of frequency ν at angle $\cos^{-1}\eta$ when the incoming radiation is of frequency ν_1 and has angle of incidence $\cos^{-1}\zeta$. This is reduced to an integral equation for a function $\varphi(x)$ of a single variable, viz.,

$$\varphi(x) = 1 + \tfrac{1}{2}x\varphi(x)\int_0^1 \frac{\varphi(x')}{x+x'}K(x')dx',$$

where $K(x')$ depends only on the absorption and scattering coefficients. He then shows that the distribution of intensity of radiation in an absorption line can be expressed in terms

of $\varphi(x)$. The integral equation is reduced to the form

$$\psi(x)a(x) = \tfrac{1}{2}\int_0^1 \frac{x'}{x'-x}K(x')\psi(x')dx',$$

and a solution is given obtained by a method of Carleman [Ark. Mat. Astr. Fys. 16, no. 26 (1922)]. The results are applied to show (i) that the assumption of incoherent scattering helps to diminish the discrepancy between theory and experiment concerning the central intensity of spectral lines, and (ii) that it can also explain the widening of spectral lines near the edge of the sun's disk.   *F. Smithies.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

✶Franke, G. **Geometrische Optik, einschliesslich Beugung und Interferenz.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 49–58. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

The author gives a short systematic summary of all the papers published in Germany during 1939–1946 in the field of geometric optics and diffraction optics. The papers are quoted in full (author, title, journal, page) and the survey is arranged under the following headings: (1) Basic ideas, (2) Seidel theory, (3) Dioptics, (4) Ray tracing, (5) Spectacles and visual aids, (6) Image quality, (7) Wave optics, (8) Miscellaneous.   *M. Herzberger* (Rochester, N. Y.).

Herzberger, M. **Light distribution in the optical image.** J. Opt. Soc. Amer. 37, 485–493 (1947).

The summary of this paper gives the impression that spot diagrams have been calculated from a polynomial approximation to the characteristic function of an optical system in such a way as to take account of the fifth- and possibly even the seventh-order aberrations. An examination of the paper itself removes this impression and reveals that the figures 3 to 7 compare photomicrographs of the actual images of a selected $f/5$ lens with spot diagrams in the calculation of which the fifth- and seventh-order errors were neglected. Part I of the paper describes a method (essentially Hamilton's) by which spot diagrams could in principle be calculated with the fifth- and seventh-order aberrations included in the analysis. Its practicability appears doubtful because of the heaviness of the calculations involved. Part II gives a brief outline of the author's diapoint theory and shows that this also could be made the basis of a spot-diagram analysis of the system. The main interest of the paper appears to the reviewer to lie in the figures 3 to 7, which provide a detailed picture of the amount and character, in the $f/5$ lens selected for investigation, of the inaccuracy introduced by disregarding the higher-order errors and using only Seidel errors, presumably with "compromise" coefficients derived from ray-traces. The value of the figures would be increased if the method of calculation of these coefficients were more precisely stated.

          E. H. Linfoot (Cambridge, England).

Di Jorio, M. **Sopra una teoria generale dell'immagine nei sistemi ottici aventi un piano di simmetria.** Ottica (N.S.) 2, 49–53 (1948).

The author has developed a formula which must be fulfilled for all the rays of a bundle if the caustic of a neighboring point is congruent to the caustic of the object

point such that homologous rays are connected by a fixed translation and rotation. In this paper the author postulates the necessity of giving a measure for the deviation from his formula. Though the author's idea is ingenious, the reviewer has found difficulties in the application of his basic formula (5). If the rays from the object point do not form a symmetric bundle, the author's formula requires the choice of a principal ray. Unfortunately, formula (5) is not independent of the choice of the principal ray and thus the measure given for the deviation contains an arbitrary factor, i.e., is not an invariant of the bundle itself. This objection does not hold if the author restricts his theory to the neighborhood of a principal ray.   *M. Herzberger.*

Di Jorio, Mario. **La condition générale pour que deux ondes-image infiniment voisines soient liées par une loi déterminée. (La loi différentielle générale pour deux ondes indépendantes).** Rev. Optique 28, 400–405 (1949).

The author rediscovers a formula which goes, in principle, back to W. R. Hamilton. Compare the reviewer's Strahlenoptik [Springer, Berlin, 1931], page 12, formula 2, where the corresponding law is shown to be valid even for an isotropic and inhomogeneous media.   *M. Herzberger.*

Grenat, Henri. **Imagerie générale du 3ᵉ ordre.** Rev. Optique 28, 389–399 (1949).

The author treats the dependency of third order aberrations on the position of the object plane. He obtains some of the results published in the reviewer's book Strahlenoptik [Springer, Berlin, 1931].   *M. Herzberger.*

Wachendorf, F. **Bestimmung der Bildfehler 5. Ordnung in zentrierten optischen Systemen.** Optik 5, 80–122 (1949).

A discussion of fifth order aberrations, which avoids the errors inherent in Schwarzschild's methods. The author computes also the fundamental relations between the aberration coefficients and computes the errors for a single surface. The composition of image errors of fifth order and their connection with third order aberration is clearly recognized and the author knows that the position of the pupil and its aberration influence the error coefficients.

          M. Herzberger (Rochester, N. Y.).

Stephan, W. G. **Decentered optical systems.** Appl. Sci. Research B. 1, 273–283 (1949).

Let S_1 and S_2 be two systems with rotational symmetry. The author considers the aberrations introduced into the system consisting of S_1 and S_2 if the two axes do not fully coincide. The difference of the two axes can be interpreted as a combination of a rotation and a translation. The author

considers the second order errors introduced by each of these defects and calculates the aberration coefficients for a system of surfaces. *M. Herzberger* (Rochester, N. Y.).

Bilimovitch, Anton D. Pfaff's method in the geometrical optics. Glas Srpske Akad. Nauka **189**, 183–191 (1946). (Serbian. English summary)

Derivation of the Hamiltonian equations in optics from Fermat's principle using vector notation. The survey of literature omits Bruns [Abh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **21**, 325–436 (1895)], and Sommerfeld and Runge [Ann. Physik (4) **35**(340), 277–298 (1911)] in which some of the material is presented in a similar way.

M. Herzberger (Rochester, N. Y.).

Richter, W. Perspektive von Spiegelungen an Dreh-flächen. Ing.-Arch. **12**, 344–363 (1941).

A historical survey of the problem is followed by a discussion of the problem of reflection of a plane bundle by a circle. The author finds three types of hyperbolas which can be used to determine the mirror images of a point. The procedure is generalized for a spherical, a cylindrical, and a conical mirror, and nomographical methods are developed to describe the perspective relations between object and image reflected by a mirror with axial symmetry. The paper contains some geometrical results which should be valuable for scientists and artists trying to paint reflections on objects with metallic surfaces. *M. Herzberger.*

Nijboer, B. R. A. The diffraction theory of optical aberrations. II. Diffraction pattern in the presence of small aberrations. Physica **13**, 605–620 (1947).

[For part I cf. Physica **10**, 679–692 (1943); these Rev. **6**, 108.] Zernicke introduced in his diffraction theory a series of polynomials deriving from hypergeometric functions which are orthogonal in the circle of unit radius. Nijboer suggests developing the characteristic function with respect to these circle polynomials and calculates the diffraction patterns for a single aberration coefficient with respect to the new development. The special coefficients connected by Nijboer with the errors known as astigmatism and coma are calculated and disagreement is found with the representation of coma diffraction given by G. C. Steward [Philos. Trans. Roy. Soc. London. Ser. A. **225**, 131–198 (1925)]. The author assumes that the disagreement might come from the fact that Steward calculated the light intensity at too small a number of points off the axis. The presentation of Nijboer's ideas is very lucid; the possibility of adding aberrations by using the circular function of Zernicke might give his investigations value even for purely geometrical optical considerations. *M. Herzberger.*

Miles, J. W. On the diffraction of an electromagnetic wave through a plane screen. J. Appl. Phys. **20**, 760–771 (1949).

This paper starts with a brief survey of some of the diffraction problems in electromagnetic theory which have been solved. The author proposes to study some of these problems further by methods used in recent work. After describing the differential equations to be solved and the boundary conditions imposed upon them, the author introduces vector transforms to represent the field vectors. For the class of problems which he considers (the incidence of a plane wave on a plane obstacle or a plane with an aperture), these transforms are written in terms of variables which mate with the coordinates of the plane of the obstacle (or

aperture). Upon imposing the electromagnetic boundary conditions for a perfect conductor, it is found that a pair of "dual" vector integral equations appear. [For the notion of dual, see E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford, 1937, pp. 334–349. The reviewer notes that the use of dual integral equations may be avoided here.] Upon giving an expression for the transmitted power and defining the aperture impedance, the author gives some variational principles for the admittance which were used by Schwinger [unpublished work]. This is followed by a brief discussion of the Babinet principle [cf. W. Watson, The Physical Principles of Wave Guide Transmission and Antenna Systems, Oxford, 1947; these Rev. **9**, 125]. Some remarks are made about the Kirchhoff results and some two-dimensional problems are discussed. *A. E. Heins.*

Honerjäger, Richard. Über die Beugung elektromagneti-scher Wellen an einem Drahtgitter. Ann. Physik (6) **4**, 25–45 (1948).

The paper deals with the diffraction of a linearly polarized monochromatic plane wave at an infinite plane grating of mutually identical evenly spaced metal wires of circular cross section. The electric field of the incident wave is parallel to the wires, which are supposed to be perfectly conducting and of diameter small compared to both the wave-length and the grid constant. Unlike Wessel [Hoch-frequenztech. Elektroak. **54**, 62–69 (1939)] the author does not confine himself to the case of normal incidence. The transmission coefficient of the grating and the intensity of the diffracted wave are calculated and presented in a number of curves, which are compared with experimental results obtained by wave-guide technique. The mathematics used by the author involves such series as $\sum_1^\infty \cos (nx) H_0^{(2)}(ny)$, where H denotes the Hankel function. Unpublished results due to H. Schmidt are cited [see, however, W. Magnus and F. Oberhettinger, Formeln und Sätze für die speziellen Funktionen der mathematischen Physik, 2d ed., Springer, Berlin, 1948, chapter III, § 13, p. 59; these Rev. **10**, 38].

C. J. Bouwkamp (Eindhoven).

Meixner, J. Theorie der Beugung elektromagnetischer Wellen an der vollkommen leitenden Kreisscheibe und am vollkommen leitenden ebenen Schirm mit kreis-förmiger Öffnung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 74–75 (1946). Cf. the following review.

Meixner, Joseph. Strenge Theorie der Beugung elektro-magnetischer Wellen an der vollkommen leitenden Kreis-scheibe. Z. Naturforschung **3a**, 506–518 (1948).

This paper is a continuation of earlier work by the author [cf. same Z. **1**, 496–498 (1946); these Rev. **9**, 124, and the preceding review]. He states that it is doubtful whether his former solution of the problem of diffraction of a plane electromagnetic wave at a circular disk has a physical meaning. The present solution is an improvement in that the author aims at solving Maxwell's equations under the condition that the electric and magnetic field vectors, though being infinite at the edge of the disk, are quadratically integrable [cf. C. J. Bouwkamp, Physica **12**, 467–474 (1946); these Rev. **8**, 363].

First the author proves the validity of the rigorous principle of Babinet, relating to mutually complementary plane diffracting obstacles. The proof is simple, and the author's result can be shown to agree with Copson's formulation [Proc. Roy. Soc. London. Ser. A. **186**, 100–118 (1946); these

Rev. 8, 179]. No reference to Copson's paper is given. Then the author considers in detail the case of a circular disk, or the complementary aperture in an infinite plane screen. As in his earlier paper, the field vectors are expressed in terms of Debye potentials, which are developed into oblate-spheroidal wave functions of integral order. The final solution is claimed to have the correct singularities at the edge of the disk. Owing to the intricacy of this solution, the reviewer has not verified all steps in the argument; it seems, however, that there now appears a singularity at the origin of the coordinate system, which should not be present in the physical solution.   *C. J. Bouwkamp* (Eindhoven).

Mirimanov, R. G. The diffraction of a spherical electromagnetic wave by a thin spherical segment. Doklady Akad. Nauk SSSR (N.S.) **67**, 65–67 (1949). (Russian)

In a previous paper [same Doklady (N.S.) **66**, 641–644 (1949); these Rev. **10**, 764] the author presented a general solution, valid to the first order in the thickness, for the reflection of electromagnetic waves from thin open surfaces of finite curvature (and uniform thickness). The method is here applied to the case of a spherical segment, with the source of the waves at the center of the sphere. The source is a point dipole directed either parallel or perpendicular to the axis of symmetry. The solution is obtained formally as an infinite series involving zonal harmonies and Bessel and Hankel functions. Mention is made of the requirement that the conductivity is finite, which was not explicitly recognized in the earlier paper.   *W. H. Furry* (Copenhagen).

Mirimanov, R. G. The diffraction of a spherical electromagnetic wave from a paraboloid of revolution of finite extent, the dipole emitting the field lying along the axis of symmetry of the paraboloid. Doklady Akad. Nauk SSSR (N.S.) **67**, 835–838 (1949). (Russian)

The author's general method [same Doklady (N.S.) **66**, 641–644 (1949); these Rev. **10**, 764] for diffraction by bodies of small constant thickness ("surfaces") and finite conductivity, previously applied to the case of a spherical segment [see the preceding review], is applied here to another special case. The formal solution in infinite series is obtained.
          W. H. Furry (Copenhagen).

Mirimanov, R. G. The diffraction of a spherical electromagnetic wave by a paraboloid of revolution of finite extent, the dipole emitting the field lying perpendicular to the axis of symmetry of the paraboloid. Doklady Akad. Nauk SSSR (N.S.) **67**, 1021–1023 (1949). (Russian)

The author has treated the problem with the dipole along the axis in an earlier paper [see the preceding review]. The field strengths for the present case are calculated in terms of derivatives of the scalar function previously defined.
          W. H. Furry (Copenhagen).

Bremmer, H. Some remarks on the ionospheric double refraction. II. Reduction of Maxwell's equations; W.K.B. approximation. Philips Research Rep. **4**, 189–205 (1949).

This is a continuation of a paper [same vol., 1–19 (1949); these Rev. **10**, 656] where the geometric-optical approximation has been dealt with. The author now works out the W.K.B. approximation, considering the field generated inside the ionosphere by a plane incident wave as the sum of four waves: the ordinary and extraordinary rising waves,

and the ordinary and extraordinary descending waves. The W.K.B. approximation becomes invalid at a height where the index of refraction times the cosine of the angle of incidence becomes imaginary. This condition characterizes the reflection levels of the ordinary and extraordinary waves. It is found by this way that such levels correspond to a horizontal direction of the ray (and not of the wave normal). The author further works out the intensity of a plane wave totally reflected by the ionosphere and applies it to the field of a transmitter by means of the well-known decomposition of the vertical dipole radiation into a sum (integral) of plane waves. The saddle-point method is then applied for finding a first approximation to this integral. The saddle-point corresponds to the geometric-optical ray that joins the transmitter with the receiver.
          G. Toraldo di Francia (Florence).

Abelès, Florin. Sur la propagation normale des ondes électromagnétiques dans les milieux stratifiés quelconques. Rev. Optique **28**, 279–287 (1949).

In a stratified medium, characterized by the two functions $\epsilon(z)$, $\mu(z)$, the normal propagation of an electromagnetic field $E = Ue^{j\omega t}$, $H = Ve^{j\omega t}$ is governed by the two equations

$$(1) \qquad \partial U/\partial z = -jk\mu V, \quad \partial V/\partial z = -jk\epsilon U.$$

Let $U = f(z)$, $V = g(z)$ and $U = F(z)$, $V = G(z)$ be two solutions of (1), satisfying the conditions $f(z_0) = 0$, $g(z_0) = 1$, $F(z_0) = 1$, $G(z_0) = 0$. Then the solution U, V, which assumes the values U_0, V_0 at $z = z_0$, is given by the matrix relation

$$\begin{pmatrix} U \\ V \end{pmatrix} = (B)^{-1} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} F & f \\ G & g \end{pmatrix} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}.$$

Now it is easy to calculate the matrix (B) for a homogeneous layer of given thickness. For a set of p different layers one obtains, by applying the continuity relations for the tangential components of the field,

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = (B_1)(B_2) \cdots (B_p) \begin{pmatrix} U_{p+1} \\ V_{p+1} \end{pmatrix}$$

and hence the reflection and transmission coefficients of the whole are readily deduced. The products of two or more matrices (B) form a group, but a single matrix (B) does not belong to this group. This is a simple demonstration of the theorem of Herpin [C. R. Acad. Sci. Paris **225**, 182–183 (1947); these Rev. **9**, 166], which states that in general a multilayer system is equivalent to two layers, but not to a single layer. The author points out the analogy of the equations (1) with those governing electric transmission lines and with the canonical equations of mechanics.
          G. Toraldo di Francia (Florence)

Eckart, G. Sur la propagation des ondes électromagnétiques "planes" dans un milieu stratifié. Recherche Aéronautique 1949, no. 8, 33–35 (1949).

The author considers the propagation of electromagnetic waves in a dielectric medium whose dielectric constant is a linear function of height, $\epsilon_r = \alpha z + \beta$, and obtains solutions of Maxwell's equations which reduce to the known plane wave solutions for horizontal or vertical polarization when $\alpha = 0$. Reflection phenomena at a boundary between free space ($\epsilon_r = 1$) and a medium of this type are also discussed.
          M. C. Gray (Murray Hill, N. J.).

Eckart, G., et Kahan, Théo. Sur le choix des chemins d'intégration dans le problème relatif au rayonnement d'un dipôle au-dessus d'un sol plan. C. R. Acad. Sci. Paris **227**, 969–970 (1948).

Kahan, T., et Eckart, G. La propagation des ondes électromagnétiques au-dessus du sol. Solution du problème de l'onde de surface. J. Phys. Radium (8) **10**, 165–176 (1949).

Kahan, T., and Eckart, G. On the electromagnetic surface wave of Sommerfeld. Physical Rev. (2) **76**, 406–410 (1949).

These three papers continue and extend an earlier note by the same authors [C. R. Acad. Sci. Paris **226**, 1513–1515 (1948); these Rev. **9**, 637]. The third is apparently an English version of the second. The reviewer maintains his critical attitude towards the authors' investigations. Whereas in the note already reviewed they accepted Epstein's solution [Proc. Nat. Acad. Sci. U. S. A. **33**, 195–199 (1947); these Rev. **9**, 126], the authors now are aware that Epstein's explanation is incorrect, thus agreeing with the reviewer's comments on Epstein's paper. They now accept Sommerfeld's original contour of integration (which they term "chemin de Weyl") and point out that Sommerfeld's evaluation of the integral along the branch cuts is in error. They are apparently unaware of earlier work of Wise [Bell System Tech. J. **16**, 35–44 (1937)] and Rice [ibid., 101–109 (1937)]. Notwithstanding their unfruitful discussions about the existence or nonexistence of the surface wave, the authors should be given credit for having stressed that the radiation condition at infinity (possibly) plays a decisive part in characterizing the true solution to the physical problem. However, the attempted proof of the uniqueness problem, formulated for real-valued wave numbers on the basis of Sommerfeld's radiation condition, breaks down: the only important conclusion that can be drawn from the authors' equation (24) of the third paper is that Ru_1 and Ru_2 tend to zero as $R \to \infty$. More powerful methods seem to be required for proving that u_1 and u_2 vanish identically. Even if their proof were considered to be correct, the authors have not realized themselves that they eventually should have shown that the solution finally accepted did satisfy Sommerfeld's radiation condition. Instead of this, they point out that the surface-wave term does not comply with this condition. *C. J. Bouwkamp* (Eindhoven).

Papas, C. H., and King, Ronold. Currents on the surface of an infinite cylinder excited by an axial slot. Quart. Appl. Math. **7**, 175–182 (1949).

The authors discuss the distribution of current-density over the surface of an infinite cylinder of radius a, fed from an axial slot of angle θ_0 over which a constant distribution of electric field is maintained. This applied field is expanded in a complex Fourier series and matched to an external TM field at the surface of the cylinder. Then the surface current density J_θ is obtained from the longitudinal magnetic intensity over the surface. The radiation pattern can also be obtained from the distant field. Curves are included showing the variation of J_θ and of the radiation pattern for $\theta_0 = 0.1$ and 0.05 radians and for various values of the radius of the cylinder. *M. C. Gray* (Murray Hill, N. J.).

Chien, W. Z., Infeld, L., Pounder, J. R., Stevenson, A. F., and Synge, J. L. Contributions to the theory of wave guides. Canadian J. Research. Sect. A. **27**, 69–129 (1949).

Part I is by Infeld, Stevenson and Synge. It is concerned with the problem of determining the field due to a source of radiation inside a rectangular wave guide, semi-infinite in length and blocked on one end by a barrier. It is assumed that the current distribution is known and that the boundaries of the wave guide, including the barrier, are perfect conductors. Three methods are employed, two based on the theory of images and the third directly on the Maxwell equations. Finally, the radiation resistance is calculated. Part II is by Stevenson and is concerned with a guide of arbitrary cross section. The calculation of the radiation resistance and reactance is discussed. Part III is by Chien and here we find calculated the radiation resistance of antennae of various shapes with assumed current distributions (for rectangular and circular guides). Part IV is by Infeld (with appendices by Pounder and Stevenson). Explicit calculations for the impedance of a linear antenna in a rectangular guide are given. *A. E. Heins.*

Pignedoli, Antonio. Moto di un elettrone in un campo magnetico e in un campo elettrico sovrapposti, uniformi ed uniformemente rotanti intorno ad un asse, con velocità angolari diverse. Atti Sem. Mat. Fis. Univ. Modena **2**, 112–115 (1948).

In an earlier paper [Atti Soc. Nat. Mat. Modena (6) **77**, 40–44 (1946); these Rev. **9**, 127] the author discussed the case when the angular velocities of the applied magnetic and electric fields were equal; in the present paper the fields still rotate uniformly but with unequal velocities. The equations of motion of an electron in such a field can be integrated by simple quadratures. *M. C. Gray.*

Zeuli, Tino. Condizioni per l'esistenza di un integrale analogo a quello delle aree nel problema del moto di un corpuscolo elettrizzato in un campo elettrico e in un campo magnetico qualsiansi sovrapposti. Atti Sem. Mat. Fis. Univ. Modena **2**, 20–36 (1948).

The author proves that it is possible to determine a first integral of the equations of motion of an electron in an electromagnetic field, $md\mathbf{v}/dt = e[\mathbf{E} + \mathbf{v} \times \mathbf{H}]$, if a vector function $\mathbf{u}$ of position can be found to satisfy the equations $\mathbf{v} \cdot d\mathbf{u}/dt = 0$, grad $U \cdot \mathbf{u} = df/dt$, $\mathbf{u} \times$ grad $V =$ grad W, where U and V are the electric and magnetic potentials, f is a function of time only and W a function of position. Then the first integral is $\mathbf{v} \cdot \mathbf{u} - (e/m)[W - f(t)] =$ constant. Conditions on the components of $\mathbf{u}$ in a general orthogonal system of coordinates are discussed, and the results applied to the case of cylindrical coordinates. When both $\mathbf{E}$ and $\mathbf{H}$ are symmetric about the z-axis two very simple first integrals can be obtained. *M. C. Gray* (Murray Hill, N. J.).

Schwinger, Julian. On the classical radiation of accelerated electrons. Physical Rev. (2) **75**, 1912–1925 (1949).

The present paper discusses the properties of radiations from very energetic accelerated electrons. In the case of electrons moving in an arbitrary trajectory, it is shown that the radiation is limited to a small cone in the forward direction with its energy centered about the critical frequency ω_c, defined by $\omega_c = \frac{3}{2}(c/R)(E/mc^2)^3$. The case in which the electrons are moving with constant speed in a circular path is also discussed. A derivation of the invariant generalization of Larmor's formula for the radiation from an electron is given. *C. Kikuchi* (East Lansing, Mich.).

Garelli, C. M., e Malvano, R. Trasformazione relativistica di onde elettromagnetiche cilindriche. Nuovo Cimento (9) **6**, 200–206 (1949).

A Lorentz transformation is applied to a cylindrical electromagnetic wave, and expressions are obtained for the

frequency, phase velocity, attenuation factor, etc., of the wave in a frame of reference which is moving in the direction of propagation.   *A. J. McConnell* (Dublin).

Quantum Mechanics

✳**Eddington, A. S. Fundamental Theory.** Cambridge, at the University Press; New York, The Macmillan Company, 1946. viii+292 pp. $6.00.

E. T. Whittaker has prepared for publication the manuscript for this book, which consists of eleven complete chapters and a part of a twelfth. A note, probably written on the last day of Eddington's working life, indicating what the continuation of the book would have been is included. There is also included as an appendix a reprint of a previously published article [Proc. Cambridge Philos. Soc. **40**, 37–56 (1944)]. The purpose of the book appears to be a development ab initio of relativistic quantum theory which takes into account not only the particular system under study but also the rest of the material in the universe and the interaction between these two. Chapters I–V and VI–VIII form two different lines of approach to "fundamental physics." In the formal sequence of deduction the latter set of chapters is supposed to precede the former. This reader was not helped by the author's deviation from the logical order.

The first five chapters constitute the "statistical theory" part of fundamental physics. In this part the author treats various "object systems" as members of a collection of similar systems and "derives" some fundamental physical constants such as the ratio of the mass of the proton to that of the electron. In the argument he uses the terminology and some of the results of standard quantum theory without proper justification. He points out on pages 3 and 4 that an enormous amount of labor is saved by making an early junction with current quantum theory and states: "The reader interested in logical rigour should bear in mind that the development of the theory turns partly on strict deduction and partly on the ultimate saving of labour. The former part requires proof, the latter part success."

The junction with current quantum theory is made incorrectly, in the opinion of the reviewer. Eddington argues that in the equations of quantum theory the coordinates must refer to a "physical origin" which is the centroid of a large number of particles. However, in quantum theory the states of a physical system are described by points in a Hilbert space of square integrable functions defined over a configuration space. The configuration space is labeled by coordinates which Eddington would call geometrical coordinates and do not refer to any physical origin. Thus current quantum theory deals with "locatable" systems and not as Eddington insists "non-locatable" ones. It may be that he is correct and that current quantum theory is wrong, but in that case he cannot use results of that theory without establishing them from his postulates. Presumably the justification for doing so is that the theory given in this book is "successful" in the sense that the values of various constants are "deduced." The measure of success is then the agreement between the calculated and observed values of these constants given on pages 66 and 105. This agreement is indeed very striking but it only proves to the reviewer that Eddington was an amazingly adept numerical manipulator and rationalizer. Eddington has not presented a logically clear and coherent physical theory from which these results follow.

Moreover, some of the "successes" of the theory are not clear cut. For example, Eddington claims that the non-Coulomb force between two protons must be Gaussian. The latest data on proton-proton scattering can be fitted equally well by a Gaussian force, a Yukawa force or a square-well. In addition Eddington proposes to explain nuclear forces without meson-fields and in fact rejects them entirely. He calculates "the mass of the mesotron" to be 173.98 electron masses. Thus his theory seemingly has no room for the various types of mesons found experimentally. However, he does state, page 214, "There seems no reason why there should not also exist 'heavy mesotrons' which decay into protons and negatons (negatively charged protons). Their mass is . . . 2.38 m_p (m_p is the mass of the proton)." No such particle has yet been found experimentally.

Chapters VI–VIII of this book are devoted to a review of sedenion analysis which had been expounded in the author's earlier book "Relativity Theory of Protons and Electrons" [Cambridge University Press, 1936]. The mathematical material contained in these chapters may be found in clearer form in various papers on four-component spinors. However, the physical interpretation of these results is quite different from that of other authors. Thus Eddington uses the known correspondence between planes of a Minkowski four-space and points of a quadric in a six-space to introduce two new dimension variables, the phase and the scale variable, and claims that these enter in the wave equations. In spite of the close connection between Lorentz transformations and inner automorphisms of sedenions, Eddington claims that the wave equations describing certain particles described by sedenions should not be Lorentz invariant. The situations for which Lorentz invariance applies and those for which it does not apply are never clearly distinguished. Chapter IX is devoted to applying the results of sedenion analysis to nuclear phenomena. In addition to rejecting meson fields, the author determines the mass defect of deuterium, the mass defect of helium, the separation constant of isobaric doublets, the mass of a neutron, and treats other topics.

In chapter X, entitled The Wave Equation, the author determines the energy of the various states of the hydrogen atom; that is, he derives the Sommerfeld formula. However, there is no indication in this or the next two chapters, which deal with the molar electromagnetic field and radiation, that the Sommerfeld formula must be corrected in order to predict the observed spectrum of hydrogen in accordance with the experiments of Lamb and Retherford. The results of these experiments, performed in 1947, were not available to Eddington. However, it does not seem to be too much to ask from a successful fundamental theory that it predict the results of future experiments correctly or at least indicate the method by which such results may be taken into account. Orthodox quantum theory has not done the latter in a completely satisfactory way but it has made some progress in that direction. It remains to be seen whether an equal or greater amount of progress can be made along the lines of Eddington's fundamental theory.

A. H. Taub (Urbana, Ill.).

Firsov, O. B. On the theory of scattering in a centrally symmetric field. Doklady Akad. Nauk SSSR (N.S.) **68**, 241–244 (1949). (Russian)

The radial factor of the Schrödinger wave-function for a state of angular momentum lh is a solution of a certain integral equation. The phase-shift appearing in the asymp-

totic form of the radial factor is expressed in terms of the integrals appearing in the equation. The solution of the integral equation by a method of iteration is equivalent to the use of Born's approximation method in solving the Schrödinger equation. This formulation provides a new way of stating conditions for the applicability of Born's method. The possibility of applying a corresponding iteration method to problems of bound states is suggested.

W. H. Furry (Copenhagen).

Izmaĭlov, S. V. On the relativistic quantum theory of particles possessing internal rotational degrees of freedom. I. The theory of particles with spin 1/2 and internal rotational motion. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 629–647 (1947). (Russian)

The author develops a relativistic quantum mechanical analogue of König's theorem that the energy of a rigid body is the sum of its energies of translation and rotation. To this end the internal motion of a particle is described by two 4×4 anti-symmetric tensors, Ω_{ij} and M_{ij}, corresponding, respectively, to angular velocity and angular momentum. A simple generalization of Dirac's equation, obtained by adding a multiple of $\Omega^{ij}M_{ij}$ to the mass, is taken as basic. Analogues of Euler's equations governing the change of M_{ij} are derived, and analogues of orbital and spin momenta, whose sum is a constant of the motion, are introduced. The author treats the case of fermions at length but indicates in a brief concluding section that a similar analysis, employing Kemmer matrices, could be carried through for bosons. A possible application of the equations studied might be the nucleus regarded as a single particle with internal rotational degrees of freedom. *A. J. Coleman* (Toronto, Ont.).

Dyson, F. J. The S matrix in quantum electrodynamics. Physical Rev. (2) **75**, 1736–1755 (1949).

The application of quantum electrodynamics to scattering problems is discussed in terms of the calculation of the S matrix, an operator which converts the ingoing waves of the initial state into the outgoing waves of the final state. For this sort of problem the Feynman formulation of the theory is well adapted. This formulation has not been published by its originator; in an earlier paper [same vol., 486–502 (1949); these Rev. **10**, 418] the author has established the general equivalence of the Feynman theory with the Tomonaga-Schwinger theory and obtained results useful in the present discussion.

The calculation of the S matrix is represented by a set of graphs, in which directed lines represent electrons and undirected dotted lines represent photons or interactions with a given electromagnetic field. Internal lines of a graph represent virtual states of particles, either serving to provide interactions between observed particles or else representing fluctuations of the fields, giving rise to effects such as self-energy; lines extending to the edge of a graph represent observed particles or interactions with the given field. There are, of course, many graphs for any given process, and corresponding to each of these there is a contribution to the S matrix, which can be written down from inspection of the graph, according to rules devised by Feynman and presented in the author's previous paper.

For a given process there is a definite set of "irreducible" graphs, which may be described roughly as having no more internal structure than required to connect the prescribed external lines. An infinite number of additional graphs can be obtained by introducing complications into the irreducible

graphs. The total effect of these complications in contributing to the S matrix can be taken into account by replacing the operators corresponding to individual lines and vertices of the irreducible graphs by modified operators whose construction is described in the author's previous paper and the present one. For present purposes the operators are considered in momentum space.

In the construction of the modified operators infinities occur, even when the number of complicating lines considered is limited. The infinities are of three kinds: (a) "accidental" infinities, appearing for special values of momenta and corresponding to simultaneous occurrence of unrelated processes; (b) "infra-red" infinities, appearing as certain momenta approach zero; and (c) "high-energy" infinities arising from integrations extending to infinite values of the momenta. The first two types are successfully avoided in certain special problems by suitable changes in the approach to perturbation calculations, and it is believed that they can be avoided in general. Infinities of the third type cannot be avoided in the existing theory, and have troubled the theory throughout its history of twenty years of successive reformulations. The paper is concerned only with these last infinities. Their removal by reinterpretation of the results has been a main feature of the advances of the last few years.

After analysing the basic internal structures which can give rise to such infinities, the author develops a systematic procedure for their complete removal, which can be extended to graphs of arbitrary complication. It is shown that the removal of all these infinities corresponds simply to a renormalization of charge and mass, and that residual finite effects are unambiguously determined. This abstract and schematic argument is admittedly not completely conclusive, it being not inconceivable that actual calculations in higher orders might reveal unforeseen difficulties. With this reservation, it is at least formally established that the existing theory, despite its unsatisfactory mathematical structure which starts with an intrinsically infinite formula (Hamiltonian) and extorts from it finite results (energies, S matrix) is, in the author's words, "no longer certainly incorrect." *W. H. Furry* (Copenhagen).

Tati, Takao, and Tomonaga, Sin-itiro. A self-consistent subtraction method in the quantum field theory. I. Progress Theoret. Physics **3**, 391–406 (1948).

In a series of earlier papers [Tomonaga, same journal **1**, 27–42 (1946); Koba and the authors, ibid. **2**, 101–116, 198–208 (1947); Kanesawa and Tomonaga, ibid. **3**, 1–13, 101–113 (1948); these Rev. **10**, 226, 227] a formally covariant quantum electrodynamics has been developed. In this paper the authors apply their formalism to the problem of the self-energy of the electron. The earlier papers and the present paper are parallel in content to Schwinger's papers [Physical Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 345] and [ibid. (2) **75**, 651–679 (1949); these Rev. **10**, 663] respectively. The authors use (independently) the same canonical transformation as Schwinger, in order to exhibit explicitly the second-order effects of the radiation field on the properties of electrons. By this transformation, the radiation field is separated into a part bound to electrons and a part behaving as a free field. The electron self-energy appears as usual as a logarithmically divergent term, which can now be identified and subtracted out of the equations of motion in an unambiguous and self-consistent way.

F. J. Dyson (Birmingham).

Belinfante, Frederik J. The interaction representation of the Proca field. Physical Rev. (2) **76**, 66–80 (1949).

The author shows by an example that the methods used by Tomonaga [Progress Theoret. Physics **1**, 27–42 (1946); Koba, Tati, and Tomonaga, ibid. **2**, 101–116 (1947); these Rev. **10**, 226] and Schwinger [same Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 345] in quantum electrodynamics can be generalized to meson theory, choosing the particular case of neutral vector mesons interacting only with the current-density four-vector of particles obeying a Dirac equation (thus no "tensor interaction"). The complications encountered are carefully listed and discussed. The analogy between the field in question and an ordinary electromagnetic field is stressed but it is expected that the same methods can be applied to other types of meson fields. It appears, however, that Schwinger's formalism must be handled with some care. *L. Hulthén* (Stockholm).

Belinfante, Frederik J. On the part played by scalar and longitudinal photons in ordinary electromagnetic fields. Physical Rev. (2) **76**, 226–233 (1949).

In the present paper, the author determines the form of the zeroth order wave functional needed in carrying through quantum electrodynamical calculations without first eliminating the Coulomb field. It is shown that the zeroth order approximation is given by a state in which "longitudinal" and "scalar" photons occur in pairs. This wave functional is then applied to the calculation of electron self-energy, electron-electron scattering, and Breit's interaction energy. *C. Kikuchi* (East Lansing, Mich.).

Hu, Ning. On the treatment of quantum electrodynamics without eliminating the longitudinal field. Physical Rev. (2) **76**, 391–396 (1949).

It is shown that consistent results can be obtained and simplification achieved in quantum electrodynamical calculations if the longitudinal component of the electromagnetic field is not first eliminated. [See also the preceding review.] *C. Kikuchi* (East Lansing, Mich.).

✶Wentzel, Gregor. Quantum Theory of Fields. Translated by Charlotte Houtermans and J. M. Jauch. With an Appendix by J. M. Jauch. Interscience Publishers, Inc., New York, N. Y., 1949. ix+224 pp. $6.00.

The German edition was published by Deuticke, Wien, 1943; Edwards, Ann Arbor, Mich., 1946; these Rev. **9**, 556.

Watson, K. M., and Jauch, J. M. Phenomenological quantum electrodynamics. III. Dispersion. Physical Rev. (2) **75**, 1249–1261 (1949).

The method of field quantization developed in earlier papers [Physical Rev. (2) **74**, 950–957, 1485–1493 (1948); these Rev. **10**, 346] is extended to include dispersive fields. The theory is applied in particular to Čerenkov radiation and it is shown that the result is in agreement with that of classical theory given by I. Tamm and I. M. Frank [C. R. (Doklady) Acad. Sci. URSS **14**, 109–114 (1937)] and I. Tamm [Acad. Sci. USSR. J. Phys. **1**, 439–454 (1939)]. *C. Kikuchi* (East Lansing, Mich.).

Groenewold, H. J. Unitary quantum electron dynamics. I, II. Nederl. Akad. Wetensch., Proc. **52**, 133–144, 226–239 (1949).

The author considers as a unitary theory one which describes the motion of a system of particles (sources) in terms of the coordinates of the particles themselves without the intervention of fields referring to other particles (carriers). A theory which refers to both sources and carriers is called a dualistic one. Current quantum electrodynamics is of the latter type. This theory is reviewed and a suggested set of equations for a unitary theory of a system of electrons is given. A decomposition of the interaction energy operator of this theory into Coulomb energy and transverse part is made. This decomposition parallels that of the dualistic theory. *A. H. Taub* (Urbana, Ill.).

Matthews, P. T. A note on Podolsky electrodynamics. Proc. Cambridge Philos. Soc. **45**, 441–451 (1949).

The physical consequences of the electrodynamics developed by Podolsky et al [cf. Podolsky and Schwed, Rev. Modern Physics **20**, 40–50 (1948); these Rev. **9**, 551] are considered. It is in particular pointed out that this theory should predict a faster rate of shower production at high energies than that given by the ordinary quantum electrodynamics. The difficulties attending the introduction of an indefinite metric and negative probabilities are indicated. [Reviewer's note: it might perhaps be pointed out that recently K. J. Le Couteur [same Proc. **44**, 63–75 (1948); these Rev. **9**, 400] has considered the problem of indefinite metric and negative probabilities from a general point of view.] *C. Kikuchi* (East Lansing, Mich.).

Van Isacker, J. Sur la construction de tenseurs symétriques d'impulsion-énergie dépendants de spineurs. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 451–456 (1949).

The author gives a general form of the field equations derived from a Lorentz invariant variational principle involving a Lagrangian function which depends on a spinor field and its derivatives. He then shows how to construct a symmetric tensor of zero divergence, the stress energy-tensor. The invariance of the variational principle under Lorentz transformations is used in an essential manner. *A. H. Taub* (Urbana, Ill.).

Tonnelat, Marie-Antoinette. Théorie unitaire du champ physique. III. Détermination des tenseurs fondamentaux. C. R. Acad. Sci. Paris **228**, 1846–1848 (1949).

The author gives a discussion of equations previously obtained [same vol., 368–370, 660–662 (1949); these Rev. **10**, 408, 498] in which second and higher order terms in one anti-symmetric tensor are neglected. *A. H. Taub*.

Pomeranchuk, I. On a generalization of the lambda-limiting process and the nonuniqueness in the removal of divergence difficulties in the quantum theory of elementary particles. Physical Rev. (2) **76**, 298–299 (1949).

A method is discussed for eliminating divergences from field theories, both classical and quantized, by a generalization of Dirac's λ-limiting process [see W. Pauli, Rev. Modern Physics **15**, 175–207 (1943); these Rev. **5**, 277]. It is shown that the method succeeds in removing all divergences, but introduces an arbitrariness into the calculation of finite observable quantities, and is therefore unsatisfactory. *F. J. Dyson* (Birmingham).

Watanabe, Satosi. Wave equations in the de Sitter space. Physical Rev. (2) **76**, 296–297 (1949).

A 4-dimensional de Sitter space-time is supposed to be imbedded in a flat 5-space. A field which satisfies a wave equation in the 5-space, and which varies in a specified way in the direction normal to the de Sitter space, will satisfy

inside the de Sitter space a wave equation which is a generalisation of the usual wave equations of flat 4-space.

F. J. Dyson (Birmingham).

Watanabe, Satosi. On Dirac's general transformation function. IIa. Progress Theoret. Physics **3**, 378–390 (1948).

The author investigates possibilities for carrying over into quantum theory the transformation function which he has previously discussed as a basis for classical field dynamics [same journal **2**, 71–88 (1947); these Rev. **10**, 227]. It is shown that the transfer to quantum theory can be made without difficulty for infinitesimal transformations, but in general not for finite transformations. Also, the method is consistent with the usual commutation-relations of quantum theory if and only if the surfaces in terms of which the transformation-function is defined are space-like. The paper is to be continued in a later issue of the same journal.

F. J. Dyson (Birmingham).

Green, Alex E. S. On generalizing boson field theories. Physical Rev. (2) **75**, 1926–1929 (1949).

The author considers the properties of Fermi particles which are coupled to a Bose field, when the Bose field possesses a generalised form of Lagrangian containing higher derivatives of the field-quantities than the first. Such a generalised Bose field has been discussed by F. Bopp [Ann. Physik (5) **38**, 345–384 (1940); these Rev. **2**, 336] and B. Podolsky [same Rev. (2) **62**, 68–71 (1942); these Rev. **4**, 31]. It is shown that for many purposes the generalised Bose field is equivalent to a mixture of ordinary Bose fields with different rest-masses. Also, the interactions between the Fermi particles produced by the generalised field are free from disagreeable singularities at short distances.

F. J. Dyson (Birmingham).

Born, M., and Green, H. S. Quantum theory of rest-masses. With appendices by K. C. Cheng and A. E. Rodriguez. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 470–488 (1949).

It is postulated that in the c-number theory the wave functions representing particles of integral spin satisfy differential equations of the form $F(p)\psi = 0$, where $F(p)$ is a general differential operator which is factorable into the form $F(p) = F_1(p)\prod_i(p_k p^k - K_i^2)$, where

$$p_k p^k = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - \partial^2/c^2 \partial t^2.$$

Corresponding to each K_i there is then a particle with a mass related to this K_i in the usual manner. For particles of spin one-half it is assumed that $F(p) = F_1(p)\prod_i(\alpha_k p^k - K_i)$, where the α_k are the Dirac matrices and $-ip_k$ are the usual differential operators with respect to the kth coordinate. The functions $F(p)$ are determined from Born's principle of reciprocity. This is used as follows. It is required that F satisfy the equation (1) $S(x, p)F = sF$, where s is a number and $S(x^k, p_k) = S(x^k, i\partial/\partial x^k) = S(p_k, i\partial/\partial p^k) = S(p_k, -x_k)$. It is claimed that for the determination of F it is sufficient to choose $S = x_k x^k + p_k p^k$ in the integral spin case and $S = \alpha_j p^j + i\gamma \alpha_j x^j$ in the case of spin one-half, where γ is the product of the four Dirac matrices. In the latter case $S(x_j, p_j) = TS(p_j, -x_j)T^{-1}$, where $T = (1+i\gamma)/(1-i)$. Once S is selected the possible masses are determined from the roots of the solutions of equation (1). These are shown to be (in the case of spin zero or one) Laguerre polynomials and rest masses close to those of the various mesons are found and others. In the case of particles of spin $\frac{1}{2}$ it is

shown that there exist particles of mass zero and others. Charged particles of mass zero are taken to be electrons. The mass of the electron is assumed to be wholly electromagnetic in origin.

The paper also contains a discussion of the determination of a Lagrangian principle for the determination of the wave equation for arbitrary $F(p)$. In addition the second quantization of the resulting field theory is given.

A. H. Taub (Urbana, Ill.).

Thermodynamics, Statistical Mechanics

Polvani, G. Il concetto di "traccia di una trasformazione" e il secondo principio della termodinamica. Rend. Sem. Mat. Fis. Milano **18** (1947), 140–173 (1948).

Ansbacher, F., and Ehrenberg, W. The derivation of statistical expressions from Gibbs' canonical ensemble. Philos. Mag. (7) **40**, 626–631 (1949).

The rigorous derivation of the distribution functions for ideal Fermi-Dirac and Bose-Einstein gases can be given from the grand canonical ensemble by elementary methods or from the canonical ensemble by means of the method of steepest descent. The authors give an elementary, rigorous derivation starting from the canonical ensemble.

L. Tisza (Cambridge, Mass.).

de Groot, Sybren R., et Tolhoek, Hendrik A. Un théorème général sur les probabilités de transition d'un système quantifié avec dégénérescence spatiale. C. R. Acad. Sci. Paris **228**, 1794–1796 (1949).

Preliminary note on a generalized proof of the theorem of spectroscopic stability and the sum rule of Ornstein and Burger, valid also for forbidden transition. *L. Tisza.*

Fényes, Imre. Zur wellenmechanischen Herleitung des statistischen Atommodells. Z. Physik **125**, 336–346 (1949).

The Thomas-Fermi statistical theory of the atom has been derived from wave mechanics by Dirac and Brillouin. The author has simplified and extended this derivation to include the refinements of the statistical theory by Amaldi, Hellmann and Weizsäcker. The relation between the various versions of the theory becomes apparent. *L. Tisza.*

Bogolyubov, N. N., and Gurov, K. P. Kinetic equations in quantum mechanics. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 614–628 (1947). (Russian)

This paper extends to quantum mechanics methods previously developed by the authors [Bogolubov, Acad. Sci. USSR. J. Phys. **10**, 257–264, 265–274 (1946); these Rev. **9**, 72; Gurov, Dissertation, Moscow, 1946]. Without using combinatory arguments, a formula is obtained for the dependence on binary central interactions of the time-rate of change of the distribution function with respect to momentum of a Bose system of identical monatomic molecules. *A. J. Coleman* (Toronto, Ont.).

Stupočenko, E. V. On the distribution of kinetic energy in systems with sources of particles. Doklady Akad. Nauk SSSR (N.S.) **67**, 635–638 (1949). (Russian)

A spatially uniform and isotropic distribution of sources of particles (as from a chemical reaction) is considered to exist throughout a volume containing a gas. The energy distribution of the source particles is assumed to be given

and constant in time, and the resultant energy distribution of the gas particles is sought. Under the assumption of (a) equality of the masses of the source and gas particles, (b) an ideal potential barrier interaction common to all pairs of particles, and (c) monokinetic source particles, the asymptotic expression (for energies large compared to kT) for the energy distribution of the gas particles is obtained by finding an approximate solution of the Boltzmann equation. The distribution obtained differs from a Maxwellian one.

$\qquad\qquad$ G. M. *Volkoff* (Vancouver, B. C.).

Miller, A. R. Statistical mechanics of high polymer solutions.
$\quad$ Australian J. Sci. Research. Ser. A. **1**, 319–329 (1948).

$\quad$ The author examines the configurational partition function for random mixtures containing any numbers of com-

ponents which can consist of simple, simple chain, branched chain, or closed ring molecules. A set of partial differential equations is obtained for the appropriate combinatory factors. Precise formulae can be deduced rigorously (a) for all binary mixtures whether the components consist of simple, simple chain, branched chain, or closed ring molecules; (b) mixtures of any number of components containing not more than one high polymer species, which can consist of closed ring equally well as of simple chain or branched chain molecules; and (c) mixtures of any number of components containing more than one high polymer species provided these consist only of simple chain or branched chain molecules.

$\qquad\qquad$ T. *Alfrey, Jr.* (Brooklyn, N. Y.).

BIBLIOGRAPHICAL NOTES

✱**Matematika v SSSR za Tridcat' Let 1917–1947.** OGIZ, Moscow-Leningrad, 1948. 1044 pp.

$\quad$ Contents: (A) Foundations of mathematics and mathematical logic, by S. A. Yanovskaya. (B) Number theory, by A. O. Gel'fond. (C) Algebra I (polynomials and fields), by N. G. Čebotarёv. Algebra II (groups, rings and lattices), by A. G. Kuroš. Topological algebra and Lie groups, by A. I. Mal'cev. (D) Topology, by A. A. Markov. (E) Descriptive set theory, by A. A. Lyapunov and P. S. Novikov. Metric theory of functions of a real variable, by N. K. Bari, A. A. Lyapunov, D. E. Men'šov and G. P. Tolstov. Polynomial approximation to functions of a real variable, by S. M. Nikol'skiĭ. Theory of functions of a complex variable, by A. F. Bermant and A. I. Markuševič. (F) Ordinary differential equations, by V. V. Nemyckiĭ and V. V. Stepanov. Partial differential equations, by S. L. Sobolev. (G) Calculus of variations, by V. V. Stepanov and L. È. Èl'sgol'c. Integral equations, by V. I. Smirnov. Functional analysis, by M. G. Kreĭn and L. A. Lyusternik. (H) Theory of probability, by B. V. Gnedenko and A. N. Kolmogorov. Mathematical statistics, by N. V. Smirnov. (I) Approximate methods, by L. V. Kantorovič and V. I. Krylov. Aids to calculation [tables, instruments, machines], by K. A. Semendyaev. Nomography, by S. V. Bahvalov. (J) Differential geometry of three-dimensional space, by P. K. Raševskiĭ. Geometry in the large, by A. D. Aleksandrov. Synthetic geometry, by S. S. Byušgens and A. A. Glagolev. (K) History of mathematics, by A. P. Yuškevič. (L) Mathematics in Latvia, by A. Ya. Lusis. (M) Mathematics in Esthonia, by A. K. Humal. Each of the main subdivisions (A)–(M) has a full bibliography, and there is an index of names.

$QA\ 3.594 \qquad QA\ 401.\ W24$

✱**Naturforschung und Medizin in Deutschland 1939–1946.**
$\quad$ Dieterich'sche Verlagsbuchhandlung, Wiesbaden. Each volume, DM 10 = \$2.40.

$\quad$ This is the German edition of FIAT Review of German Science 1939–1946, Office of Military Government for Germany, Field Information Agencies Technical, from which it differs only in substituting German for English title pages and tables of contents and the omission of a page of introductory matter. The articles of both editions are in German, and will be reviewed separately in Mathematical Reviews. Volumes containing articles of mathematical interest which have appeared so far are as follows: 1, 2, Reine Mathe-

matik I, II; 5, Angewandte Mathematik III (Mathematische Grundlagen der Strömungsmechanik); 6, Angewandte Mathematik IV (Geodäsie); 7, Angewandte Mathematik V; 8, 9, Physik der festen Körper, I, II; 17, 18, Geophysik, I, II; 20, Astronomie, Astrophysik und Kosmogonie.

Annals of the Institute of Statistical Mathematics.

$\quad$ The Institute is located in Tokyo. Volume **1**, no. 1 is dated August, 1949.

Bulletin of the Technical University of Istanbul.

$\quad$ Volume **1**, no. 1 is dated 1948. The main title is in Turkish, İstanbul Teknik Üniversitesi Bülteni.

Communications de la Faculté des Sciences de l'Université d'Ankara.

$\quad$ Volume **1** is dated 1948. This is stated to be a foreign-language edition of Ankara Üniversitesi Fen Fakültesi Mecmuası.

Proceedings of the Japan Academy.

$\quad$ This continues Proceedings of the Imperial Academy (Tokyo). Numbers 3–10 of volume **21** (1945) were issued in 1949.

Rad Jugoslavenske Akademije Znanosti i Umjetnosti.

$\quad$ In view of changes in title and inconsistencies in numbering volumes, the following detailed information seems desirable. In 1940 appeared Rad Jugoslavenske Akademije Znanosti i Umjetnosti, Knjiga **267**, Razreda Matematičko-Prirodoslovnoga 83. In the main title, Jugoslavenske was changed to Hrvatske in volumes **271**, **274** and **278**, which appeared as numbers 84, 85, and 86 under the subtitle in 1941, 1942, and 1945; the latter numbers were used as volume numbers in Mathematical Reviews. In 1948 appeared Rad Jugoslavenske Akademije Znanosti i Umjetnosti, Knjiga **271**, Odjel za Matematičke, Fizicke i Tehničke Nauke, Knjiga I. The contents are not the same as those of the earlier Knjiga **271**. Foreign-language summaries of the contents of the later Knjiga **271** are contained in Bulletin International de l'Académie Yougoslave des Sciences et des Beaux-Arts, Classe des Sciences Mathématiques, Physiques et Techniques, Livre **2** (1949), which is stated to be a continuation of Livre **35**, Classe des Mathématiques et Sciences Naturelles (1945).

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